UNIVERSITÉ DE MONS
FACULTÉ DES SCIENCES

Carrollian fermions coupled to gravity

Lea Mele

Master's thesis presented in partial fulfillment of the requirements for the degree of

Master in Physics.

Service de Physique de l'Univers, Champs et Gravitation

Supervisor: Andrea Campoleoni

Academic year 2022-2023

Acknowledgements

First and foremost, I would like to express my deep gratitude to my supervisor, Andrea Campoleoni, for his guidance and his support all along the year. I am truly grateful for the thorough reviews and feedback I received during the writing of this master's thesis.

Then, I would like to thank Simon Pekar for the invaluable discussions we had about this project during the second semester and for the feedback provided during the writing of this master's thesis.

Additionally, I would like to thank William Delplanque, Robin Guarini, Delio Sollai and Sylvain Thomée with whom I have spent the year, doing my master's thesis in the Physique de l'Univers, Champs et Gravitation unit.

Furthermore, I am grateful to my friends for their unwavering encouragement, Helena, my friend of many years, and the group "Les Sangs", with Floriane, Pauline, Romane, Guillaume L., Émilien, Guillaume D. and Eduardo.

I would especially like to thank my boyfriend, his encouragement has been a constant source of motivation throughout this journey.

Finally, I would also like to thank my parents, my sister Noa, my brother Teo, and all my family for their support.

Contents

Introduction

This thesis is dedicated to the study of the so called Carroll limit of field theories, with a particular focus on those including fermionic matter.

The Carroll limit is the limit of vanishing speed of light. It was first mentioned in 1965 by Jean-Marc Lévy-Leblond in [1], where he introduced the "Carroll group" as a "degenerate cousin" of the Poincaré group, obtained from the latter by a group contraction where the speed of light is sent to zero. The name refers to Lewis Carroll, mathematician and author of the famous book *Alice's Adventures in Wonderland* [2], and its sequel *Through the Looking-Glass* [3]. As explained in his paper, the purpose of the definition of the Carroll group was "mainly pedagogical", and it was introduced as a mathematical curiosity [4]. And as suspected by its own author, this limit remained relatively unnoticed for over thirty years. This non-infatuation is understandable considering the occurrence of various peculiar phenomena in this limit, such as the fact that the light cone collapses to a line, meaning that causality almost completely disappears in a "Carrollian" universe. These kinds of mathematical peculiarity are captured in the novel *Through the Looking-Glass*,

"Well, in our country," said Alice, still panting a little, "you'd generally get to somewhere else if you run very fast for a long time, as we've been doing." "A slow sort of country!" said the Red Queen. "Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!".

We can interpret this dialogue as the Queen presenting to Alice the "Carrollian world". In this world, Carrollian observers are moving at the speed of light, which is the maximal speed, and are immobile at the same time[a]. With the lack of causality and dynamics in this limit, it seems difficult to make practical use of this Carroll group. However, against all odds, since the 2000s, more and more papers considering the Carrollian limit have appeared on subjects like general relativity, field theory, etc. Indeed, in recent years, various applications of Carrollian physics have emerged in interesting physical contexts, such as holography, black holes, gravitational waves, or cosmology (see, e.g., [6, 7, 8, 9]). To illustrate how a limit where the speed of light is sent to zero can be used in concrete physical contexts, let us provide some details at its application to holography, which has recently revived interest in Carrollian physics.

The holographic principle states that in a gravitational theory defined in $D+1$ space-time dimensions, all the dynamics of this theory can be encoded in a quantum field theory defined

[a]The lack of causality in the novels of Lewis Carroll inspired J.-M. Lévy Leblond for the name of the limit (c.f. chapter 7, "A Mad Tea-Party" [2]). Using this dialogue between Alice and the Queen as an illustration, F. Dyson ([5]) further justified this name using the immobility of the Carrollian observers. This explanation is inspired by [6].

on a D dimensional space-time [10]. A concrete realization of this holographic principle is the AdS/CFT correspondence. This correspondence consists of a holographic *duality* between a gravitational theory in $D+1$ dimensions, with a negative cosmological constant, and a conformal field theory (CFT) in D dimensions. In recent years, a great deal of research has been carried out into the extension of a holographic correspondence without a cosmological constant, i.e., for an asymptotically flat space-time. An approach that makes possible to define a holography without cosmological constant is called "Carrollian holography". It exploits the fact that in the AdS/CFT correspondence, the limit of vanishing cosmological constant on the gravitational "side" corresponds to a Carroll limit of vanishing speed of light on the CFT "side" of the duality. In this limit, the conformal group reduces to the conformal extension of the Carroll group (see, e.g., [7]).

Given its application to a multitude of physical contexts, each more interesting than the last, we now know why it is worth studying the Carroll limit. The literature on bosonic field theories in the Carrollian limit is already quite extensive (see, e.g., [11] and references therein). However, little attention has been paid to fermions (see however [12, 13] for two earlier attempts, and [14]). Nevertheless, fermions are a main part of the Standard Model of Particle physics. All the matter around us is made up of fermions, hence the need to include them in the study of Carrollian field theories. This is the principal motivation of this thesis.

Concretely, the aim of this thesis is twofold: Firstly, we explore the possible inequivalent Carrollian limits of the massless spin-1/2 fermionic field theory in a Hamiltonian formulation, along the lines of the systematic analysis of bosonic theories performed in [11]. In a second part, we plan to couple the massless Dirac fermions to Carrollian gravity, thus extending to matter couplings the analysis of [15]. The goal of the latter paper was to clarify the links between the Hamiltonian analysis of [11] and the definition of Carrollian gravity given in [16] through a gauging of the Carroll algebra.

Recently, it was shown that there exists not one but two inequivalent Carrollian limits for each bosonic Lorentz-invariant theories [11]. The theories obtained in the limit are called "electric" and "magnetic" theory. Both theories can be obtained from the Hamiltonian formulation of the relativistic theory by sending the speed of light to zero, but with two different rescalings of the canonical variables. The two limits are worked out for all bosonic field theories, from the simplest examples, such as scalar field or electromagnetism, to full Einstein's gravity and the free theory describing higher-spin particles. However, half-integer spin fields are not investigated. In this thesis, we therefore extend this analysis to the case of Dirac fermionic field theory.

The second part of the original contribution of this work consists in the coupling of Dirac fermions to gravity in the Carrollian limit. In a recent study [15], it has been proven that the magnetic limit of Einstein's gravity defined through the Hamiltonian formalism in [11] is equivalent to the Carrollian action obtained in [16] through a gauging of the Carroll algebra. The problem was also studied in the following way. Rather than directly examining the outcomes of the Carrollian limit, this result has been revisited in [15], using the relation between the Hamiltonian and the first-order formulations of gravity. In this thesis, we extend this analysis by adding the coupling of massless Dirac fermions to Carrollian gravity.

We start in chapter 1 by recalling some useful concepts and by setting out the conventions used in this thesis. In particular, we present the Carroll and Galilei geometries and algebras obtained via the $c \to 0$ and $c \to \infty$ limits, and discuss the physical implications of these limits. This parallel between the Galilean and Carrollian limits is often realized in the literature because the Carrollian limit can be seen as a counterpart of the more familiar, non-relativistic, $c \to \infty$ limit. Pursuing with this idea of duality between the two limits in chapter 2, we review the Galilean and Carrollian limits of different theories that can be found in the literature, with a special focus on the Dirac theory. Using what we have learned about the Carrollian and Galilean limits of the Dirac theory from this review, and following the analysis of [11], we study the Carrollian limits of the Dirac action in its Hamiltonian form. This constitutes the first part of the original contribution of this thesis, devoted to the study of the free theory of Dirac fermions in the Carrollian limit. In chapter 3, a short review of the Hamiltonian formulation of general relativity is realized. Finally, in chapter 4, we start by recalling some concepts required to describe the coupling of gravity to spinors fields, and we then used this notion as well as the Hamiltonian formulation of general relativity in the investigation of the Carrollian limit of gravity coupled to fermions. This is the second original contribution of our work, dedicated to the study of Carrollian gravity coupled to Dirac fermions. Finally, we summarize our results and discuss the future possible directions of this work.

Chapter 1

Bases and conventions

The aim of this chapter is to recall and define some of the basic concepts and conventions used throughout this thesis. As the Hamiltonian formalism and the spin-1/2 fermions will be used extensively, we begin with a few basic reminders about these notions. Then, to better understand the limit of vanishing speed of light, we give a short review of the Carrollian and Galilean limits.

1.1 Conventions

In this master's thesis, we work in a four-dimensional space-time and the "mostly" plus convention $(-,+,+,+)$ for the signature of the metric is used. The Greek indices correspond to space-time indices, Latin indices $i, j, \ldots$ take their values in $\{1,2,3\}$, and spinor indices are not explicitly written. The symmetric and antisymmetric parts of a certain tensor $A_{\mu\nu}$ are respectively defined in our conventions as $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$ and $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$. Moreover, units where $\hbar = 1$ are used. However, when certain general concepts are recalled, we will also set $c = 1$. This will be explicitly stated in such cases. Powers of c will then be reintroduced by dimensional analysis if necessary.

1.2 Hamiltonian formalism

Let us begin with some brief recalls about the Hamiltonian formulation of classical mechanics and field theory based on the books of E. Poisson "A relativist's toolkit" [17], and "Field Quantization" by W. Greiner and J. Reinhardt [18]. We choose the units where $c = 1$ in this particular section, with $x^\mu = (x^0, x^i) = (t, x^i)$.

1.2.1 Mechanics

In the Lagrangian formulation of classical mechanics, one is given a Lagrangian $L(q, \dot{q})$, function of the generalized coordinate q and its velocity

$$\dot{q} \equiv \frac{dq}{dt}. \tag{1.1}$$

By integrating the Lagrangian over a selected path $q(t)$, an action functional $S[q]$ can be constructed
$$S[q] = \int_{t_1}^{t_2} dt\ L(q, \dot{q}). \tag{1.2}$$
The equations of motion, known as the Euler-Lagrange equations, are obtained by varying the action such as it is stationary, i.e. such that $\delta S = 0$, where the variation is restricted by $\delta q(t_2) = \delta q(t_1) = 0$. One can use an equivalent formulation, the Hamiltonian formulation, by introducing the canonical momentum
$$p \equiv \frac{\partial L}{\partial \dot{q}}, \tag{1.3}$$
as an independent variable instead of the velocity $\dot{q}$. It is assumed that this relation can be inverted to give $\dot{q}$ as a function of p and q. We introduce then the Hamiltonian via the *Legendre transformation*
$$H(q,p) = p\dot{q}(p) - L\left(q, \dot{q}(p)\right). \tag{1.4}$$
Hamilton's equations of motion, which are the Hamiltonian equivalent of the Euler-Lagrange equations, can be derived from the variation of the action with respect to p and q independently, under the restriction that $\delta q(t_1) = \delta q(t_2) = 0$. Requiring the action to be stationary, Hamilton's equations take the form
$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}. \tag{1.5}$$
Finally, recall that the Poisson bracket of two dynamical variables $A(p,q)$ and $B(p,q)$ is defined as
$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}. \tag{1.6}$$

1.2.2 Field theory

The above description deals with systems characterized by a discrete set of coordinates $q^i(t)$. The following content describes, in a compact manner as well, the Hamiltonian formulation of field theories. Let us consider a generic field $\phi(x)$. The Lagrangian function is now a functional of the field $\phi(x)$,
$$L(t) = \int d^3x\ \mathcal{L}(x). \tag{1.7}$$
To switch to Hamilton's formalism, one needs to define the canonically conjugate field, the "momentum". In analogy to (1.3), it is defined as
$$\pi(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}(x)}, \quad \text{with } \dot{\phi}(x) \equiv \frac{\partial \phi(x)}{\partial t}, \tag{1.8}$$
where $\mathcal{L}$ is the Lagrangian density. The Hamiltonian is introduced in the same way as in the mechanics case through the Legendre transformation
$$H(t) = \int d^3x\ \pi(x)\dot{\phi}(x) - L(t). \tag{1.9}$$

It can also be written in terms of the Hamiltonian density $\mathcal{H}(x)$

$$H(t) = \int d^3x \ \mathcal{H}(x), \tag{1.10}$$

where $\mathcal{H}(x) = \pi(x)\dot{\phi}(x) - \mathcal{L}(x)$. The equivalent version of Hamilton's equations (1.5) for a field theory, where we omit from now on the dependence of the coordinates of the functional and the fields, is the *functional derivative* of the functional H with respect to π and ϕ

$$\dot{\phi} = \frac{\delta H}{\delta \pi}, \quad \dot{\pi} = -\frac{\delta H}{\delta \phi}. \tag{1.11}$$

Since the Hamiltonian can depend on the field ϕ, its conjugate π and their gradients $\nabla\phi$, $\nabla\pi$, Hamilton's equations can be expressed in terms of the Hamilton density as

$$\begin{aligned} \frac{\delta H}{\delta \phi} &= \frac{\partial \mathcal{H}}{\partial \phi} - \nabla \frac{\partial \mathcal{H}}{\partial(\nabla\phi)} \\ \frac{\delta H}{\delta \pi} &= \frac{\partial \mathcal{H}}{\partial \pi} - \nabla \frac{\partial \mathcal{H}}{\partial(\nabla\pi)}. \end{aligned} \tag{1.12}$$

Finally, the Poisson brackets for fields theories is defined, given two functionals $F[\phi, \pi]$ and $G[\phi, \pi]$, as

$$\{F, G\} = \int \mathrm{d}^3x \left(\frac{\delta F}{\delta \phi(x)} \frac{\delta G}{\delta \pi(x)} - \frac{\delta F}{\delta \pi(x)} \frac{\delta G}{\delta \phi(x)} \right). \tag{1.13}$$

1.3 Spin-1/2 action

In four space-time dimensions, the four gamma matrices γ^μ define the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\ \eta^{\mu\nu}. \tag{1.14}$$

Using the four gamma matrices, it is possible to define the matrix γ^5 as

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{1.15}$$

which satisfies $\{\gamma^5, \gamma^\mu\} = 0$ and $(\gamma^5)^2 = \mathbb{1}_{4\times4}$.

The action for spin-1/2 Dirac massive fields is the Dirac action

$$S = \int d^4x\ \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi, \tag{1.16}$$

where Ψ is a four-component Dirac spinor and $\bar{\Psi}$ its Dirac conjugate, defined as

$$\bar{\Psi} = \Psi^\dagger\gamma^0. \tag{1.17}$$

From the action (1.16), the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\Psi = 0 \tag{1.18}$$

can be derived.

In this thesis, the Pauli-Dirac representation of the Clifford algebra is used repetitively. It is given by

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{1.19}$$

where σ^i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.20}$$

1.4 Carrollian and Galilean limits

It is well known that the Galilean group can be obtained from a $c \to \infty$ "non-relativistic" limit of the Poincaré group. As said in the introduction of this thesis, in the sixties, the opposite $c \to 0$ "ultra-relativistic" limit of the Poincaré group was explored for the first time by Jean-Marc Lévy-Leblond with the definition of a "degenerate" cousin of the Poincaré group, named the "Carroll group" [1]. In this section, we first look at the Galilean and Carrollian limits of the Minkowski metric. Then, we show how these two groups can be obtained from a contraction of the Poincaré algebra. Finally, we examine how the Carrollian limit affects the light cone and explain the physical implications of this phenomenon. Although we compare the two limits, we pay particular attention to the Carrollian limit, for which we provide more details. The Galilean limit presented here is to be considered as a comparison for the Carrollian limit. The content of this section is based on [19], [20] and [21].

1.4.1 Limits of the Minkowski metric

The usual *covariant* Minkowski metric reads

$$G = -dx^0 \otimes dx^0 + \delta_{ij} dx^i \otimes dx^j, \tag{1.21}$$

and the *contravariant* metric is given by

$$G^{-1} = -\frac{\partial}{\partial x^0} \otimes \frac{\partial}{\partial x^0} + \delta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}. \tag{1.22}$$

In the $c \to \infty$ or $c \to 0$ limit, we have to keep track of the powers of c in the action, and we therefore define the time coordinate

$$t = x^0/c. \tag{1.23}$$

In terms of these new coordinates (t, x^i), the covariant and contravariant metric reads

$$G = -c^2 dt \otimes dt + \delta_{ij} dx^i \otimes dx^j, \tag{1.24}$$

and

$$G^{-1} = -\frac{1}{c^2} \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} + \delta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}. \tag{1.25}$$

Galilean metric

Let us start with the familiar $c \to \infty$ limit of the Minkowski metric. The latter is obtained by letting $c \to \infty$ in the *contravariant* metric, giving the degenerate metric

$$G^{-1} \to \delta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}. \tag{1.26}$$

Carrollian metric

Using the *covariant* Minkowski metric and performing the opposite ultra-relativistic limit $c \to 0$ limit, one obtains

$$G \to \delta_{ij} dx^i \otimes dx^j. \tag{1.27}$$

Let us recall that the Poincaré group can be defined, in the appropriated coordinate system, as the group of linear transformations that leave the Minkowski metric invariant. As in the Carroll case one deals with a degenerate metric, this definition needs to be extended. In flat space and in appropriate coordinates (t, x^i), the Carroll group can be defined [15] as the group of linear transformations that leave invariant both the degenerate metric

$$(\zeta_{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} \tag{1.28}$$

and the vector

$$(n^\nu) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \zeta_{\mu\nu} n^\nu = 0. \tag{1.29}$$

1.4.2 Contractions of the algebra

Since non-relativistic mechanics is a limiting case of relativistic mechanics, the Galilei group must be in some sense a limiting case of the Poincaré group. This is the idea behind the Inönu-Wigner *group contraction* introduced in [22] by E. Inönu and E.P. Wigner (see also [23]). This group contraction for the Galilean group was extended to the Carroll group by Lévy-Leblond. In this subsection, we will see what is meant by a “group contraction” in the context of the Carroll and Galilean groups.

To define the two different contractions of the Poincaré algebra, we start from a differential realization of its generators. This choice allows us to specifically highlight their reliance on the time coordinate, and therefore on the speed of light, in order to take the corresponding limit [21].

Poincaré algebra

The differential realization of the generators of the Poincaré group is given by

$$M_{\mu\nu} = (x_\mu \partial_\nu - x_\nu \partial_\mu), \quad P_\mu = -\partial_\mu, \tag{1.30}$$

with $M_{\mu\nu}$ and P_μ the generators of Lorentz rotations and space-time translations, respectively, which obey the Poincaré algebra:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\rho} M_{\sigma\nu} + \eta_{\nu\sigma} M_{\rho\mu} - \eta_{\mu\sigma} M_{\rho\nu} - \eta_{\nu\rho} M_{\sigma\mu}, \\
[M_{\mu\nu}, P_\rho] &= \eta_{\rho\nu} P_\mu - \eta_{\rho\mu} P_\nu, \\
[P_\mu, P_\nu] &= 0.
\end{aligned} \tag{1.31}$$

We will not consider the vanishing commutation relations of the time translations generators in the following. In the coordinates (t, x^i), the generators take the form

$$\begin{aligned}
&M_{ij} = (x_i \partial_j - x_j \partial_i), \quad M_{0i} = \left(-ct\partial_i - x_i \frac{1}{c}\partial_t\right), \\
&P_i = -\partial_i, \quad P_0 = -\frac{1}{c}\partial_t.
\end{aligned} \tag{1.32}$$

The Poincaré algebra, with respect to this decomposition in time and spatial coordinates, gives

$$\begin{aligned}
&[M_{ij}, M_{kl}] = \delta_{ik} M_{\ell j} + \delta_{jl} M_{ki} - \delta_{il} M_{kj} - \delta_{jk} M_{li}, \\
&[M_{ij}, M_{0l}] = \delta_{jl} M_{0i} - \delta_{il} M_{0j}, \quad [M_{ij}, P_l] = \delta_{lj} P_i - \delta_{li} P_j, \\
&[M_{ij}, P_0] = 0, \quad [M_{0i}, P_0] = P_i, \\
&[M_{0i}, P_j] = \delta_{ji} P_0, \quad [M_{0i}, M_{0j}] = M_{ij}.
\end{aligned} \tag{1.33}$$

Carroll algebra

The Carroll algebra is obtained from a group contraction of the Poincaré algebra with the speed of light c sent to zero. Starting with the commutator of the Lorentz boosts M_{0i}, and considering the explicit expression of the generators (1.32) gives

$$\left[-ct\partial_i - x_i \frac{1}{c}\partial_t, -ct\partial_j - x_j \frac{1}{c}\partial_t\right] = M_{ij}. \tag{1.34}$$

Then, multiplying by c^2 provides

$$\left[-c^2 t\partial_i - x_i \partial_t, -c^2 t\partial_j - x_j \partial_t\right] = c^2 M_{ij}, \tag{1.35}$$

and the $c \to 0$ limit gives

$$[-x_i \partial_t, -x_j \partial_t] = 0. \tag{1.36}$$

By performing a rescaling with c of the generators of the Poincaré boosts $cM_{0i} \equiv B_i$, one obtains, in the $c \to 0$ limit, the commutation relation of the Carrollian boosts

$$[B_i, B_j] = 0, \tag{1.37}$$

where $B_i = -x_i \partial_t$. Considering now the commutator of P_0 and M_{0i}

$$\left[-ct\partial_i - x_i \frac{1}{c}\partial_t, -\frac{1}{c}\partial_t\right] = P_i, \tag{1.38}$$

multiplied by c^2 and in the $c \to 0$ limit, one obtains

$$[-x_i\partial_t, -\partial_t] = 0. \tag{1.39}$$

By performing the same rescaling by c for the boost and the time translations generators, $cM_{0i} \equiv B_i$ and $cP_0 \equiv H$, and then taking the $c \to 0$ limit, the commutation relation for the Carrollian boosts and time translations reads

$$[B_i, H] = 0, \tag{1.40}$$

with $B_i = -x_i\partial_t$ and $H = -\partial_t$. By performing this analysis for the other commutation relations, the Carroll algebra is obtained:

$$\begin{array}{ll} \multicolumn{2}{c}{[M_{ij}, M_{kl}] = \delta_{ik}M_{\ell j} + \delta_{jl}M_{ki} - \delta_{il}M_{kj} - \delta_{jk}M_{li},} \\ [M_{ij}, B_l] = \delta_{jl}B_i - \delta_{il}B_j, & [M_{ij}, P_l] = \delta_{lj}P_i - \delta_{li}P_j, \\ [M_{ij}, H] = 0, & [B_i, H] = 0, \\ [B_i, P_j] = \delta_{ji}H, & [B_i, B_j] = 0, \end{array} \tag{1.41}$$

with M_{ij}, P_i, H, B_i which are respectively the generators of spatial rotations, spatial translations, time translations, and Carrollian boosts. We have therefore shown that the Carrollian algebra is obtained as the algebra in the $c \to 0$ contraction of the Poincaré algebra. One can see that in the Carrollian case, with respect to the Poincaré algebra, the boosts commute with each other and with the time translations.

With the help of the degenerate metric (1.29), the above commutation relations can be written compactly as follows

$$\begin{aligned} [M_{\mu\nu}, P_\rho] &= \zeta_{\rho\nu}P_\mu - \zeta_{\rho\mu}P_\nu \\ [M_{\mu\nu}, M_{\rho\sigma}] &= \zeta_{\mu\rho}M_{\sigma\nu} + \zeta_{\nu\sigma}M_{\rho\mu} - \zeta_{\mu\sigma}M_{\rho\nu} - \zeta_{\nu\rho}M_{\sigma\mu}. \end{aligned} \tag{1.42}$$

Galilean algebra

The same reasoning as above can be applied to obtain the Galilean algebra from the Poincaré algebra, but this time with the $c \to \infty$ limit and the rescaling $\frac{1}{c}M_{0i} \equiv C_i$ and $cP_0 \equiv E$ of the generators. With this group contraction, the Galilean algebra

$$\begin{array}{ll} \multicolumn{2}{c}{[M_{ij}, M_{kl}] = \delta_{ik}M_{\ell j} + \delta_{jl}M_{ki} - \delta_{il}M_{kj} - \delta_{jk}M_{li},} \\ [M_{ij}, C_l] = \delta_{jl}C_i - \delta_{il}C_j, & [M_{ij}, P_l] = \delta_{lj}P_i - \delta_{li}P_j, \\ [M_{ij}, E] = 0, & [C_i, E] = P_i, \\ [C_i, P_j] = 0, & [C_i, C_j] = 0, \end{array} \tag{1.43}$$

is obtained in the $c \to \infty$ limit. In these relations, M_{ij}, P_i, E, C_i are respectively the generators of spatial rotations, spatial translations, time translations, and Galilean boosts. The Galilean boosts are given by $C_i = -t\partial_i$. One can see that this time, the Galilean boosts commute with each other and with the spatial translations.

1.4.3 Physical interpretation

Finally, to conclude this short review about the Carrollian and Galilean limits, we try to provide an intuitive illustration of what occurs in the $c \to \infty$ and $c \to 0$ limits. Let us consider the Minkowski space-time, with its light cone (c.f. figure 1.1). In the space-time coordinates (x^0, x^i), the light cone is defined with the equation

$$-(x^0)^2 + \sum_i (x^i)^2 = 0, \tag{1.44}$$

or equivalently

$$x^0 = \pm \|\vec{x}\|. \tag{1.45}$$

Considering the two-dimensional case, and the time coordinate

$$t = x^0/c, \tag{1.46}$$

one obtains

$$t = \pm x/c. \tag{1.47}$$

In the $c \to \infty$ limit, we therefore see with this latter equation that one has a horizontal asymptote. The cone merges with the x^i plane, or equivalently, the light cone "opens", and time becomes absolute. In contrast, in the ultra-relativistic $c \to 0$ limit, one obtains a vertical asymptote. The light cone "closes". As the light cone determines the causality structure in Minkowski space-time, this closing of the light cone will rule Carrollian invariant field theories. In particular, it implies that, as the speed of light is the absolute speed and therefore one cannot go beyond the light cone, neighboring points do not speak with each other [11]. We will retrieve this behavior by studying concrete examples of Carroll-invariant field theories in subsection 2.1.2.

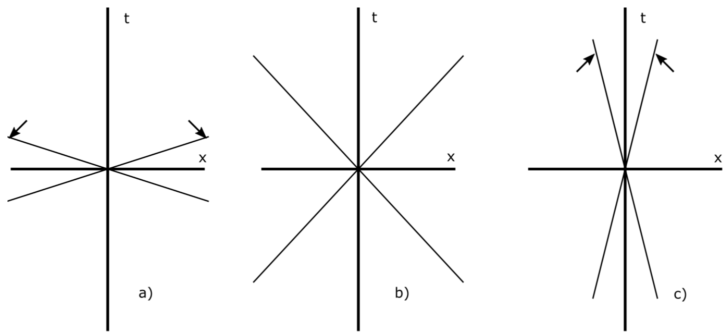

Figure 1.1: Minkowski b), Galilean a) and Carrollian c) light cone (figure inspired by [1])

Chapter 2

Limits of the free theory

In [11], it has been shown that there exist two inequivalent Carrollian limits for each Lorentz-invariant bosonic theories, the "electric" and "magnetic" limit. Each limit can be obtained from the Hamiltonian form of the corresponding theory, with the same "contraction" procedure of taking the speed of light to zero, but with two different c-dependent rescaling of the canonical variables. In this chapter, we aim to study the possible[a] inequivalent $c \to 0$ limits of free Dirac fermions. To obtain these limits, we draw upon the analysis presented in [11]. Before that, we discuss some results already present in the literature about the Carrollian and Galilean limits of fields theories, and we contrast them with the approach followed in [11].

2.1 Inequivalent Carrollian and Galilean limits of bosonic theories

Fifty years ago, it was shown that there exist not one but two well-defined non-relativistic $c \to \infty$ limits of Maxwell's electromagnetism, referred to as electric and magnetic limits [24]. The first limit is obtained when the electric effects dominate, while the magnetic limit is valid when the magnetic effects are dominant. It also has been proved that there exists the Carrollian analogue of this phenomenon for electromagnetism in four spacetime dimensions, with the existence of two Carroll-invariant contractions of electromagnetism, also referred to as "electric" and "magnetic" contractions [19]. The aim of [11] was to show that the existence of two inequivalent Carrollian limits is not specific to electromagnetism but exists for all Lorentz-invariant theories. Before reviewing the results of [11], we firstly give an overview of the two Galilean limits of classical electromagnetism. Then, we start our examination of the analysis presented in [11], where we first discuss the conditions for a theory to be Carroll invariant, and we then illustrate the analysis with the examples of electromagnetism and the scalar field.

2.1.1 Galilean electromagnetism

Let us consider two inertial reference frames R' and R, with R' moving with a constant velocity v along the axis x with respect to R. These two frames are related by Lorentz trans-

[a] As there exist two inequivalent Carrollian limits for each bosonic Lorentz-invariant theories, we expect two Carrollian theories for free Dirac fermions.

formations. For a four-vector $(u^0, \vec{u})$ these transformations take the form

$$\begin{cases} u'^0 = \gamma \left(u^0 - \dfrac{1}{c} \vec{v} \cdot \vec{u} \right) \\ \vec{u}' = \gamma \left(\vec{u} - \dfrac{1}{c} \vec{v}\; u^0 \right), \end{cases} \tag{2.1}$$

where $\vec{v} = v\vec{1_x}$ and γ is the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}}. \tag{2.2}$$

The usual Lorentz transformation of a generic four-vector above admits two different Galilean limits $v/c << 1$,

$$\begin{cases} u'^0 = u^0 \\ \vec{u}' = \vec{u} - \dfrac{1}{c}\; \vec{v}\; u^0, \end{cases} \tag{2.3}$$

valid when $|\vec{u}| << |u^0|$, and

$$\begin{cases} u'^0 = u^0 - \dfrac{1}{c} \vec{v} \cdot \vec{u} \\ \vec{u}' = \vec{u}, \end{cases} \tag{2.4}$$

for four-vectors such as $|\vec{u}| >> |u^0|$ [24]. An example for the first case is given by the usual Galilean transformation of the four-vector $(c\Delta t, \Delta\vec{r})$

$$\begin{cases} c\Delta t' = c\Delta t \\ \Delta\vec{r}' = \Delta\vec{r} - \vec{v}\Delta t. \end{cases} \tag{2.5}$$

As v is the velocity of the referential R' seen from the referential R, one has that $|\vec{v}|/c = |\Delta r/c\Delta t| << 1$ in the Galilean limit. Starting from the Lorentz transformation (2.1), considering the Galilean limit $v/c << 1$ and that it implies $|\Delta r| << c|\Delta t|$, one retrieves (2.5), corresponding to the limit (2.3).

Now, to obtain the two limits of Maxwell's equations, one can apply (2.3) and (2.4) to the case of the current four-vector $(c\rho, \vec{j})$. Let us first investigate the case where $c|\rho| >> |\vec{j}|$. In fact, this case is equivalent to the situation where $E >> cB$ and one can therefore directly study the Galilean limit of the Lorentz transformations of the electromagnetic fields. This limit will give the Galilean *electric limit*. In [24], Maxwell's equations in presence of sources are considered. This allows one to further clarify the physical interpretations of the different limits. For simplicity, we only consider here the vacuum case, which already shows the two inequivalents Galilean limits. We now see how the two limits impact the Maxwell's equations

$$\begin{cases} \boldsymbol{\nabla} \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\ -\boldsymbol{\nabla} \times \mathbf{B} + \dfrac{1}{c^2} \dfrac{\partial \mathbf{E}}{\partial t} = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{E} = 0. \end{cases} \tag{2.6}$$

In the Galilean limit $v/c << 1$ and considering that $E >> cB$, the Lorentz transformations of

E and **B**

$$\begin{cases} \mathbf{E}'/c = \gamma(\mathbf{E}/c + \frac{1}{c}\vec{v} \times \mathbf{B}) \\ c\mathbf{B}' = \gamma(c\mathbf{B} - \frac{1}{c}\vec{v} \times \mathbf{E}), \end{cases} \tag{2.7}$$

reduce to

$$\begin{cases} \mathbf{E}' = \mathbf{E} \\ \mathbf{B}' = \mathbf{B} - \frac{1}{c^2}\vec{v} \times \mathbf{E}. \end{cases} \tag{2.8}$$

In the moving frame, an observer sees a time-varying field. In the above equations, one can deduce that while this time-variation of the electric field produces a magnetic field (with the term "$\frac{1}{c^2}\vec{v} \times \mathbf{E}$"), the time-variation of the magnetic field does not produce an electric field. The time derivative of the magnetic field term in Maxwell's equations (2.6) can no longer be present and therefore read

$$\begin{cases} \boldsymbol{\nabla} \times \mathbf{E} = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\ -\boldsymbol{\nabla} \times \mathbf{B} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{E} = 0. \end{cases} \tag{2.9}$$

By a similar discussion of that of the electric limit but considering $c|\rho| << |\vec{j}|$, or equivalently $E << cB$, the *magnetic limit* of the Maxwell's equations is given by:

$$\begin{cases} \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\ \boldsymbol{\nabla} \times \mathbf{B} = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{E} = 0, \end{cases} \tag{2.10}$$

where the time-derivative of the electric field is no longer present. There therefore exist two completely different Galilean limits of Maxwell's equations, whether the electric or magnetic effects are dominant. The magnetic contraction gives an electromagnetism theory where the displacement current is missing from Ampère's law, but Faraday's law is complete. On the contrary, Ampère's law is complete in the electric limit, but the Faraday's law misses the time-varying magnetic field.

This idea of two limits for the Galilean limit of Maxwell's electromagnetism has been extended to the Carrollian limit in [19] of this theory, and then generalized to all Lorentz-invariant bosonic theories in [11] as we discuss in the following subsection.

2.1.2 Carrollian limit

It is now interesting to discuss the fact that the two limits of classical electromagnetism can be found with a direct limit and without involving arguments concerning the dominance of one of the two fields. Indeed, starting with the vacuum Maxwell's equations (2.6) and directly taking the non-relativistic limit $c \to \infty$, one arrives to the Galilean magnetic limit (2.10) of the Maxwell equations. Performing a rescaling of the fields $B = B'/c$ and $E = E'c$, and only then taking the limit, one gets the electric limit (2.9). This way of retrieving the two inequivalent Galilean limits is investigated in the case of the dual Carrollian limit in [19]. Moreover, it was shown in [11] that the existence of two inequivalent Carroll limits which differ by a c-dependent rescaling of the fields can be extended to all the bosonic Lorentz-invariant theories. These two limits are named electric and magnetic as well, whether or not there is an electromagnetic duality in the theory. This is the subject of this subsection.

We begin by discussing the conditions for Carroll invariance established in [11]. Before reviewing the Carrollian limits of electromagnetism, we continue this section with the example of the scalar field also investigated in [11]. This example is useful to define the two Carrollian limits and their characteristics thanks to its simplicity. Finally, we conclude with the case of interest in this comparison between the inequivalent Galilean and Carrollian limits: classical electromagnetism.

Conditions for Carroll invariance

In [11], one can find conditions for the theory found in the limit to be Carroll invariant. We firstly describe these conditions.

In the canonical formalism, a Carroll transformation is generated by

$$a^0 H + a^k P_k + b_k B^k + \frac{1}{2}\omega_{km} M^{km}, \quad \omega_{km} = -\omega_{mk}, \tag{2.11}$$

where the parameters b_k, ω_{km}, a^0 and a^k parameterize respectively the infinitesimal Carroll boosts, spatial rotations, time translations and spatial translations. The generators of time and space translations read[b]

$$H = \int d^3x\ \mathcal{E}(x), \qquad P_k = \int d^3x\ \mathcal{P}_k(x), \tag{2.12}$$

and those of Carroll boosts and spatial rotations are given by

$$B^k = \int d^3x\ x^k \mathcal{E}(x), \qquad M^{rs} = \int d^3x\ (x^r \delta^{sk} - x^s \delta^{rk}) \mathcal{P}_k(x), \tag{2.13}$$

where $\mathcal{E}(x)$ and $\mathcal{P}_k(x)$ are respectively the "energy density" and the "momentum density". For a theory to be Carroll invariant, a necessary and sufficient condition is that the generators obey the Carroll algebra,

$$[P_k, B^m] = \delta_k^m H, \tag{2.14}$$

$$[P_k, M^{rs}] = \left(\delta_k^r \delta^{sl} - \delta_k^s \delta^{rl}\right) P_l, \quad \left[B^k, M^{rs}\right] = -B^r \delta^{sk} + B^s \delta^{rk}, \tag{2.15}$$

$$\left[M^{km}, M^{rs}\right] = -\delta^{kr} M^{ms} + \delta^{mr} M^{ks} + \delta^{ks} M^{mr} - \delta^{ms} M^{kr}, \tag{2.16}$$

where the other Poisson brackets vanish. The requirement to fulfil the Carroll algebra implies conditions on the Poisson bracket of $\mathcal{P}_k(x)$ and $\mathcal{E}(x)$, as all the generators are constructed in terms of them. The non-trivial conditions for the theory to be Carroll invariant are, in fact, on $\mathcal{E}(x)$. There are two of them: (1) $\mathcal{E}(x)$ has to be a scalar under spatial translations and rotations; (2) $\{\mathcal{E}(x), \mathcal{E}(x')\} = 0$. The derivation of this result can be found in appendix A.

[b] In this thesis, we are in a four-dimensional spacetime. However, note that the case of Lorentz-invariant bosonic theories presented here can be extended to an arbitrary number of dimensions, as it is done in [11].

Scalar field

The usual Minkowski metric reads $ds^2 = -(dx^0)^2 + \sum_k (dx^k)^2$. In the $c \to 0$ limit, we have to keep track of the powers of c in the action and we therefore use a time variable t that has dimension of time. The metric then reads $ds^2 = -c^2 dt^2 + \sum_k (dx^k)^2$, which gives $\eta_{tt} = -c^2$ (and for the inverse component $\eta^{tt} = -\frac{1}{c^2}$). The action in Hamiltonian form for a scalar field in Minkowski space is

$$S\left[\varphi, \pi_\varphi\right] = \int dt \left[\int d^3x \; \pi_\varphi \dot{\varphi} - H\right], \quad \dot{\varphi} \equiv \partial_t \varphi, \tag{2.17}$$

with

$$H = \int d^3x \; \mathcal{E}, \quad \mathcal{E} = \frac{1}{2}\left[c^2 \left(\pi_\varphi\right)^2 + \partial_k \varphi \partial^k \varphi\right], \tag{2.18}$$

and

$$\pi_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\dot{\varphi}}{c^2}. \tag{2.19}$$

The magnetic limit is the direct $c \to 0$ limit of the above action

$$S^M\left[\varphi, \pi_\varphi\right] = \int dt \left[\int d^3x \; \pi_\varphi \dot{\varphi} - H^M\right] \tag{2.20}$$

with

$$H^M = \int d^3x \; \mathcal{E}^M, \quad \mathcal{E}^M = \frac{1}{2}\left[\partial_k \varphi \partial^k \varphi\right]. \tag{2.21}$$

One can get the electric limit by rescaling $\varphi = c\varphi'$, $\pi_\varphi = \frac{1}{c}\pi'_\varphi$ in (2.17), and only then taking the $c \to 0$ limit. Dropping the primes, this gives

$$S^E\left[\varphi, \pi_\varphi\right] = \int dt \left[\int d^3x \; \pi_\varphi \dot{\varphi} - H^E\right], \tag{2.22}$$

with

$$H^E = \int d^3x \; \mathcal{E}^E, \quad \mathcal{E}^E = \frac{1}{2}\pi_\varphi^2. \tag{2.23}$$

Notice that the rescaling of the field and of the conjugate momentum to obtain the latter limit is chosen such that the kinetic term $\int dt \int d^3x \; \pi_\varphi \dot{\varphi}$ is preserved. As explained in the previous section, the conditions for the theory to be Carroll invariant are conditions on the Poisson brackets of $\mathcal{E}$. For the case of the scalar field, the two contractions are Carroll invariant since for each case, the energy density $\mathcal{E}(x^k)$ obeys $\left\{\mathcal{E}(x^k), \mathcal{E}(x'^k)\right\} = 0$ and is a scalar under spatial translations and rotations.

The equations of motions of the magnetic and electric contractions are respectively,

$$\dot{\varphi} = 0, \quad \dot{\pi}_\varphi = \Delta \varphi, \tag{2.24}$$

and

$$\dot{\varphi} = \pi_\varphi, \quad \dot{\pi}_\varphi = 0, \quad \Rightarrow \ddot{\varphi} = 0. \tag{2.25}$$

One can notice here that the dynamical feature that distinguishes the magnetic theory from the electric one is that the momentum conjugate can be eliminated from the equation of motion in the electric case, while it cannot in the magnetic case. These equations reduce to $\varphi(t,x^k) = \varphi(0,x^k)$ and $\pi_\varphi(t,x^k) = t\Delta\varphi(0,x^k)+\pi_\varphi(0,x^k)$ for the magnetic theory, and $\pi_\varphi(t,x^k) = \pi_\varphi(0,x^k)$ and $\varphi(t,x^k) = \pi_\varphi(0,x^k)t + \varphi(0,x^k)$ for the electric theory. One can see that the fields at a certain time and spatial point depend only on the fields at $t = 0$ and at the same spatial point. This observation is compatible with the fact that in a Carrollian theory, due to the closing of the light cone, information does not propagate between neighboring points.

In [11], the authors conclude that there are two ways of producing a Carroll-invariant theory. The magnetic theory is obtained by drooping the time derivatives (conjugate momenta) and keeping the spatial gradients in the energy density. The electric limit is obtained by doing the opposite, i.e. keeping the time derivatives and dropping the spatial gradients. Now that we have reviewed the simple case of the scalar field, let us investigate these two inequivalent Carrollian limits in the case of Maxwell's electromagnetism.

Electromagnetism

The action in Hamiltonian form for electromagnetism in Minkowski space is[c]

$$\begin{aligned} S\left[A_i,\pi^i,A_t\right] &= \int dt \left[\int d^3x\ \pi^i \dot{A}_i - H\right], \\ H = \int d^3x \left(\mathcal{E} - A_t\partial_i\pi^i\right), &\quad \mathcal{E} = \frac{1}{2}\left[c^2\pi^i\pi_i + \frac{1}{2}F^{ij}F_{ij}\right], \end{aligned} \tag{2.26}$$

where F_{ij} is the spatial field strength (the magnetic field) defined as $F_{ij} = \partial_i A_j - \partial_j A_i$, and the momentum conjugate to A^i, π_i, is given by

$$\pi_i = F_{ti} = \dot{A}_i - \partial_i A_t. \tag{2.27}$$

The magnetic limit of electromagnetism is obtained via the direct $c \to 0$ limit

$$S^M\left[A_i,\pi^i,A_t\right] = \int dt \left[\int d^3x\ \pi^i \dot{A}_i - H^M\right], \tag{2.28}$$

with

$$H^M = \int d^3x \left(\mathcal{E}^M - A_t\partial_i\pi^i\right), \quad \mathcal{E}^M = \frac{1}{4}F^{ij}F_{ij}, \tag{2.29}$$

so that the energy density $\mathcal{E}^M$ verifies the condition

$$\left\{\mathcal{E}^M\left(x^k\right), \mathcal{E}^M\left(x'^k\right)\right\} = 0. \tag{2.30}$$

The equations of motions are given by

$$\begin{aligned} \delta\pi^i: &\quad F_{ti} \equiv \dot{A}_i - \partial_i A_t &= 0, \\ \delta A_i: &\quad \dot{\pi}^i - \Delta A^i + \partial^i\partial_j A^j &= 0, \\ \delta A_t: &\quad \partial_i\pi^i &= 0. \end{aligned} \tag{2.31}$$

[c] Note that electromagnetism is a gauge theory. This implies in particular that in its Hamiltonian formulation, unlike the case of the scalar field, one finds a Lagrange multiplier, A_t. This is not a dynamical variable, and therefore there is no conjugate momentum associated with it. As this section is simply to illustrate the magnetic and electric theory of this theory, we won't go into further detail.

One can rewrite these equations of motion in terms of the electric field $E^i = -\pi^i$ and the magnetic field $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk} = \epsilon^{ijk}\partial_j A_k$ as

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \quad \frac{\partial \boldsymbol{E}}{\partial t} - \boldsymbol{\nabla}\times\boldsymbol{B} = 0, \quad \frac{\partial \boldsymbol{B}}{\partial t} = 0. \tag{2.32}$$

This is precisely the magnetic Carrollian limit of the Maxwell's equations obtained in [19], and that we have discussed at the beginning of this subsection.

As in the case of the scalar field theory, the electric theory can be reached out by a rescaling of the fields, which in the case of electromagnetism, takes the form $A_i = cA'_i$, $\pi^i = \frac{1}{c}\pi^{i\prime}$, and $A_t = cA'_t$. Performing this rescaling of the fields in the initial action, taking the $c \to 0$ limit and dropping the primes, one obtains

$$S^E\left[A_i, \pi^i, A_t\right] = \int dt \left[\int d^3x\ \pi^i \dot{A}_i - H^E\right], \tag{2.33}$$

where

$$H^E = \int d^3x \left[\mathcal{E}^E - A_t\partial_i\pi^i\right], \quad \mathcal{E}^E = \frac{1}{2}\pi^i\pi_i, \tag{2.34}$$

with

$$\left\{\mathcal{E}^E(x^k), \mathcal{E}^E(x'^k)\right\} = 0. \tag{2.35}$$

The field equations are now given by

$$\begin{aligned} \delta\pi^i :& \quad \dot{A}_i - \partial_i A_t - \pi_i &= 0,\\ \delta A_i :& \quad \dot{\pi}^i &= 0,\\ \delta A_0 :& \quad \partial_i\pi^i &= 0. \end{aligned} \tag{2.36}$$

They can be rewritten in terms of the rescaled electric and magnetic fields as

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \quad \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla}\times\boldsymbol{E} = 0, \quad \frac{\partial \boldsymbol{E}}{\partial t} = 0, \tag{2.37}$$

which is the same electric limit of the Maxwell's equations found in [19]. One can find in these two Carrollian limits of electromagnetism all the characteristics highlighted in the case of the scalar field: the magnetic theory is obtained by dropping the time derivatives (π^i) and keeping the spatial gradients (F^{ij}) in the energy density; the electric theory is obtained by keeping the time derivatives and dropping the spatial gradients in the energy density; the conjugate momentum associated with A_i can be eliminated using its own equation of motion in the electric case, but not in the magnetic case. More details about these two limits can be found in [11]. Since the aim of this example is to show the Carroll equations of motions dual to the Galilean ones that we have discussed, we will restrict ourselves to this analysis.

2.2 Limits of the free Dirac fermion theory

Following the review of the Galilean limits of Maxwell's electromagnetism presented in the subsection 2.1.1, we first review the non-relativistic limit of the Dirac action that can be found in the literature to identify the key features that we will also encounter in the Carrollian limit.

Then, we discuss the "ultra-relativistic" limit of the Dirac equation obtained in [14]. Finally, we move on to the analysis of the possible inequivalent Carrollian limits of the Dirac action.

2.2.1 Non-relativistic limits of the Dirac equation

As discussed in the subsection 2.1.1, the non-relativistic limit is considering the speed of particles negligible with respect to the speed of light. Equivalently, it can also be considered as the limit where the rest energy $E_0 = mc^2$ is the dominant contribution to the energy. In the case of the Dirac action, the result of the limit is the Pauli equation, which describes a non-relativistic spin-1/2 particle. This limit can be found in many textbooks on quantum field theory, such as [25]. In this section, we would like to obtain this result from a "$c \to \infty$" limit. First, we start by recalling the limit of the massive Dirac equation in the regime where the speeds of particles are significantly smaller than c. Then, we derive it from the massive Dirac equation with the explicit $c \to \infty$ limit. Finally, we compare with the massless case already studied in [26] by an equivalent non-relativistic limit that we define in the associated subsection.

"Usual" non-relativistic limit of the massive Dirac equation

One starts from the massive Dirac equation written in the non-relativistic notation

$$i\dot{\Psi} = c\alpha^k p_k \Psi + mc^2 \beta \Psi, \quad \dot{\Psi} \equiv \partial_t \Psi, \tag{2.38}$$

where $p_k = -i\partial_k$, with $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, and the use of the Pauli-Dirac representation

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ \alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \tag{2.39}$$

where β and α^k are linked to the gamma matrices (1.19) with $\beta = \gamma^0$ and $\alpha^k = \gamma^0\gamma^k$. With this separation of the field into two-component spinors, the equation (2.38) reads[d]

$$i \begin{pmatrix} \dot{\phi} \\ \dot{\chi} \end{pmatrix} = c\vec{\sigma} \cdot \vec{p} \begin{pmatrix} \chi \\ \phi \end{pmatrix} + mc^2 \begin{pmatrix} \phi \\ -\chi \end{pmatrix}. \tag{2.40}$$

One may recall that stationary state solutions of the Dirac equation are defined as

$$\Psi(r,t) \equiv \Psi(r) e^{-iEt}. \tag{2.41}$$

Based on the latter solutions, let us assume that the fields ϕ and χ take the form

$$\begin{pmatrix} \phi \\ \chi \end{pmatrix} \equiv e^{-imc^2 t} \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix}, \tag{2.42}$$

where $\tilde{\phi}$ and $\tilde{\chi}$ are slowly varying functions of time since the rest energy, and therefore the dominant part of the time dependence, is in the exponential in (2.42)[e]. With this redefinition

[d] We use the notion $c\vec{\sigma} \cdot \vec{p} \equiv c\sigma^k p_k$ in the following.

[e] This change of variable is made such as one has the splitting of the energy $E = \Delta E + mc^2$. One can therefore see that the main contribution, mc^2, is in the exponential.

of the fields, one gets the system of two equations

$$i\begin{pmatrix}\dot{\tilde{\phi}}\\ \dot{\tilde{\chi}}\end{pmatrix} = c\vec{\sigma}\cdot\vec{p}\begin{pmatrix}\tilde{\chi}\\ \tilde{\phi}\end{pmatrix} - 2mc^2\begin{pmatrix}0\\ \tilde{\chi}\end{pmatrix}. \tag{2.43}$$

As (2.42) are stationary state solutions of the Dirac equation, one has that the time derivative of the redefined fields $\tilde{\phi}$ and $\tilde{\chi}$ gives $i\dot{\tilde{\chi}} = \Delta E\tilde{\chi}$ with $\Delta E = E - mc^2$. In the non-relativistic limit, this difference is small compared to the rest energy mc^2. We can therefore neglect the term $i\dot{\tilde{\chi}}$ in the lower equation of (2.43). The system of equations reduces then to

$$i\dot{\tilde{\phi}} = c\vec{\sigma}\cdot\vec{p}\tilde{\chi}, \quad 2mc\tilde{\chi} = \vec{\sigma}\cdot\vec{p}\tilde{\phi}. \tag{2.44}$$

Inserting

$$\tilde{\chi} = \frac{\vec{\sigma}\cdot\vec{p}}{2mc}\tilde{\phi} \tag{2.45}$$

in the first equation, gives

$$i\dot{\tilde{\phi}} = \frac{(\vec{\sigma}\cdot\vec{p})^2}{2m}\tilde{\phi}, \tag{2.46}$$

which is the Pauli equation describing a non-relativistic spin-1/2 particle[f]. One has thus demonstrated that by considering the speeds of particles negligible with respect to the speed of light, and equivalently that the rest energy $E = mc^2$ is the principal contribution to the energy, one retrieve the Pauli equation as the non-relativistic limit of the Dirac equation.

Non-relativistic $c \to \infty$ limit of the massive Dirac equation

Now we would like to find the same result as in the previous section concerning the non-relativistic limit of the Dirac equation, this time not considering the rest energy as the dominant contribution in the energy, but considering c sent to infinity. To realize this limit, we were inspired by the non-relativistic limit of the Dirac equation proposed in the book “Structural aspects of Quantum Field Theory”, by G. Grensing [27]. For this purpose, a dimensionless parameter λ such that $c = \tilde{c}\lambda$ is introduced and $\tilde{c}$ is set to 1. Sending the speed of light to infinity therefore corresponds to taking the $\lambda \to \infty$ limit. As in the “usual” non-relativistic limit, we start from the Dirac equation written in the non-relativistic notation (2.38), but with c replaced with the parameter λ. We use the same Pauli representation of the gamma matrices, as well as the same decomposition of the field into two-component spinors. Drawing on the previous section, we perform the redefinition of the fields

$$\begin{pmatrix}\phi\\ \chi\end{pmatrix} \equiv e^{-im\lambda^2 t}\begin{pmatrix}\tilde{\phi}\\ \tilde{\chi}\end{pmatrix} \tag{2.47}$$

which implies the Dirac equations

$$i\begin{pmatrix}\dot{\tilde{\phi}}\\ \dot{\tilde{\chi}}\end{pmatrix} = \lambda\vec{\sigma}\cdot\vec{p}\begin{pmatrix}\tilde{\chi}\\ \tilde{\phi}\end{pmatrix} - 2m\lambda^2\begin{pmatrix}0\\ \tilde{\chi}\end{pmatrix}. \tag{2.48}$$

[f] The Pauli equation is usually found in the literature coupled to an external electromagnetic field. The latter being irrelevant in our case, we omit it.

Note that the lower component reads

$$i\dot{\tilde{\chi}} - \lambda\vec{\sigma}\cdot\vec{p}\tilde{\phi} + 2m\lambda^2\tilde{\chi} = 0. \tag{2.49}$$

Now, rather than considering the limit where the rest energy is the dominant contribution and using the same argument as the previous subsection, we take the $\lambda \to \infty$ limit. If the limit is taken directly, the two last terms go to infinity. A solution to get rid of these divergences is to realize a rescaling of the fields[g]

$$\tilde{\chi} \to \frac{\tilde{\chi}}{\lambda}, \quad \tilde{\phi} \to \tilde{\phi}. \tag{2.50}$$

After this rescaling, the equation (2.49) takes the form

$$i\frac{\dot{\tilde{\chi}}}{\lambda} - \lambda\vec{\sigma}\cdot\vec{p}\tilde{\phi} + 2m\lambda\tilde{\chi} = 0. \tag{2.51}$$

The first term tending to zero in the $\lambda \to \infty$ limit, the two last terms must cancel each other:

$$\vec{\sigma}\cdot\vec{p}\tilde{\phi} = 2m\tilde{\chi}. \tag{2.52}$$

Applying the rescaling of the fields (2.50) on the upper equation of (2.48)

$$i\dot{\tilde{\phi}} = \lambda\vec{\sigma}\cdot\vec{p}\tilde{\chi}, \tag{2.53}$$

the λ is cancelled and the non-relativistic equations finally reduces to

$$2m\tilde{\chi} = \vec{\sigma}\cdot\vec{p}\tilde{\phi}, \quad i\dot{\tilde{\phi}} = \vec{\sigma}\cdot\vec{p}\tilde{\chi}. \tag{2.54}$$

By isolating the field $\tilde{\chi}$ in the first equation

$$\tilde{\chi} = \frac{\vec{\sigma}\cdot\vec{p}}{2m}\tilde{\phi} \tag{2.55}$$

and substituting it into the second equation of (2.54), one retrieves the Pauli equation

$$i\dot{\tilde{\phi}} = \frac{(\vec{\sigma}\cdot\vec{p})^2}{2m}\tilde{\phi}. \tag{2.56}$$

One can therefore see that the limit $c \to \infty$ and the one considers in the previous section result in the same equation.

Galilean limit of the massless Dirac equation

In both limiting procedures described above, the mass term in the Dirac action played an important role. On the other hand, the Galilean limit of the massless Dirac equation was investigated in [26]. This limit is based on the fact that the contraction of the Poincaré algebra $c \to \infty$, which gives the Galilean algebra that we have discussed in section 1.4, can be achieved

[g] This shorter way of presenting the rescaling means, in the same way as we have done in subsection 2.1.2, that we have defined $\tilde{\chi}'$ and $\tilde{\phi}'$ such as $\tilde{\chi} = \frac{\tilde{\chi}'}{\lambda}$, $\tilde{\phi} = \tilde{\phi}'$ and that we have then dropped the primes after the limit. We will present the rescaling in this manner from now on.

by setting $c = 1$ and performing the rescaling of the coordinates

$$x^k \to \epsilon\, x^k, \quad t \to t, \quad \epsilon \to 0, \tag{2.57}$$

with ϵ, a dimensionless parameter. One can also see this in the following way. In the limit $\epsilon \to 0$ and with (2.57), the equation describing the light cone in 2 dimensions obtained in section 1.4 with $c = 1$, $t = \pm x$, gives a horizontal asymptote in the $\epsilon \to 0$ limit, and this limit is therefore interpreted as a non-relativistic limit.

The analysis starts with the massless Dirac equation of motion

$$i\gamma^\mu \partial_\mu \Psi = 0 \tag{2.58}$$

with the fermionic field decomposed into two two-components spinors ϕ, and χ, $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$. Using the Pauli-Dirac representation (1.19) of the gamma matrices and setting $c = 1$, the equations of motion are given by

$$i\dot{\phi} + i\sigma^k \partial_k \chi = 0, \quad i\dot{\chi} + i\sigma^k \partial_k \phi = 0. \tag{2.59}$$

The non-relativistic equations are obtained by scaling the spinors as

$$\phi \to \phi, \quad \chi \to \epsilon\chi, \tag{2.60}$$

together with the scaling of space-time (2.57). The scaling $\phi \to \epsilon\phi$, $\chi \to \chi$ is also possible as the equations (2.59) are symmetric under the exchange of ϕ and χ. This would give the same result, but with the role of χ and ϕ exchanged. Note that the scaling of the two-components spinors (2.60) with ϵ, is equivalent to the one we introduced in the previous section with λ, such as we have $\lambda = 1/\epsilon$, with $\lambda \to \infty$ and $\epsilon \to 0$.

With the scaling of the fields, the massless Dirac equations in the Galilean limit defined by (2.57) are

$$i\sigma^k \partial_k \phi = 0, \quad i\dot{\phi} + i\sigma^k \partial_k \chi = 0. \tag{2.61}$$

The link with the $c \to \infty$ limit of the previous section can be done by taking the massless limit of the equations (2.54)

$$\vec{\sigma} \cdot \vec{p}\phi = 0, \quad i\dot{\phi} = \vec{\sigma} \cdot \vec{p}\chi, \tag{2.62}$$

such as with $p_k = -i\partial_k$, we retrieve the same equations as in [26] that we have discussed here.

To summarize, in this section we were able to study the Galilean limit of the Dirac action by different approaches. This allows us to check that the "usual" non-relativistic limit where one considers the speeds of the particles are significantly lower than the speed of light, the Galilean $c \to \infty$ limit and the one obtained by the scaling of space-time are all equivalent. This also enabled us to recover the usual Galilean limit of the Dirac equation in a way that is more adapted to study its Carrollian dual.

2.2.2 Ultra-relativistic limit of the massless Dirac equation

Now that we checked that the "usual" non-relativistic limit of the Dirac equation is consistent with the one obtained by the scaling of space-time discussed in [26], we can move to the Carrollian equivalent of this limit investigated in [14].

This limit is based on the fact that, as for the Galilean case (2.57), the contraction of the Poincaré algebra $c \to 0$ which gives the Carroll algebra that we have discussed in section 1.4 can be achieved by a scaling of space-time. For the ultra-relativistic case, this scaling is given by

$$x^k \to x^k, \quad t \to \epsilon\, t, \quad \epsilon \to 0. \tag{2.63}$$

Once again, as for the Galilean case, we can also see this in the following way. In the limit $\epsilon \to 0$, and with (2.63), the equation describing the light cone in 2 dimensions obtained in section 1.4 with $c = 1$, $t = \pm x$, gives a vertical asymptote in the $\epsilon \to 0$ limit, and this limit is therefore interpreted as an ultra-relativistic limit.

One starts with the massless Dirac equation, with Ψ decomposed into two two-components spinors and the Pauli-Dirac representation of the gamma matrices. This gives the relativistic equations of motion

$$i\dot{\phi} + i\sigma^k \partial_k \chi = 0, \quad i\dot{\chi} + i\sigma^k \partial_k \phi = 0. \tag{2.64}$$

Performing the rescaling of the fields

$$\phi \to \phi, \quad \chi \to \epsilon\chi \tag{2.65}$$

as well as the scaling of space-time (2.63), the ultra-relativitic equations of motion are

$$i\dot{\phi} = 0, \quad i\dot{\chi} + i\sigma^k \partial_k \phi = 0. \tag{2.66}$$

As in the previous section, one could have taken the scaling $\phi \to \epsilon\phi$, $\chi \to \chi$. This would have only inverse the role of ϕ and χ. These equations are the Carrollian Dirac massless equations. We will refer to them in the following section. Note that in the literature, we have not found a Carrollian limit of the massive Dirac equation[h]. A field redefinition, such as the one we have carried out in the non-relativistic limit,

$$\begin{pmatrix} \phi \\ \chi \end{pmatrix} \equiv e^{-im\lambda^2 t} \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix}, \tag{2.67}$$

is not as obvious as in the non-relativistic case, and so we restrict ourselves to the massless case. We hope to return to this issue in the future.

2.2.3 Possible inequivalent Carrollian limits of the Dirac action

Now that we are familiar with the analysis of [11] reviewed in section 2.1 and with the ultra-relativistic limit of the Dirac equation, we will seek the Carrollian contraction of the Dirac

[h] This observation is not unexpected in view of the growing interest in the Carrollian limit due to its application to flat-space holography. Indeed, the theory of Dirac fermions in the massless case is invariant under conformal transformations. This invariance is expected for Carrollian theories to obtain applications in flat-space holography, as we discussed in the introduction.

action in the Hamiltonian formalism. For a first approach, we chose to focus on the massless case.

First attempt for the Carrollian limit

The massless Dirac Lagrangian with explicit powers of c is given by

$$\mathcal{L} = \frac{i}{c}\bar{\Psi}\gamma^0\dot{\Psi} + i\bar{\Psi}\gamma^k\partial_k\Psi. \tag{2.68}$$

This action is already of first order but, to simplify the comparison with [11], we explicitly introduce the conjugate momentum to the field Ψ and rewrite it in Hamiltonian form. The conjugate momentum to the field Ψ is actually proportional to $\bar{\Psi}$ and reads

$$\pi_\Psi \equiv \frac{\partial\mathcal{L}}{\partial(\dot{\Psi})} = \frac{i}{c}\bar{\Psi}\gamma^0. \tag{2.69}$$

The action in Hamiltonian form for a Dirac fermion field in Minkowski space-time is

$$S\left[\Psi,\pi_\Psi\right] = \int dt\left[\int d^3x\ \pi_\Psi\dot{\Psi} - H\right], \tag{2.70}$$

$$H = \int d^3x\ \mathcal{E}, \quad \mathcal{E} = -c\pi_\Psi\gamma^0\gamma^k\partial_k\Psi, \tag{2.71}$$

which vanishes in the direct $c \to 0$ limit. Since in this case, $\mathcal{E}$ is also linear in π and Ψ as the kinetic term, the alternative limit introduced for bosons in [11] and that we have discussed in the subsection 2.1.2, in which one rescales the fields while leaving the kinetic term invariant also leads to the same results. Although Carroll invariant, these limits are rather unexciting from an interpretative point of view.

Carrollian limit with two-component spinors

The failure to define non-trivial limits of the Dirac action applying the same strategy as in [11] to the whole field Ψ is not unexpected, since in our previous analysis of the Carrollian limit of the Dirac equation we had to decompose Ψ. Therefore, inspired by the decomposition of the Dirac spinor into two-components spinors χ and ϕ of the Carrollian and Galilean limits of the Dirac equation presented in the previous sections, we now try to obtain a Carrollian limit of the massless Dirac action with a non-vanishing Hamiltonian.

We start from the massless Dirac Lagrangian, perform our analysis in the Pauli-Dirac representation of the gamma matrices, and decompose the field into two-component spinors $\Psi = \begin{pmatrix}\phi \\ \chi\end{pmatrix}$. The Dirac conjugate $\bar{\Psi}$ is given by

$$\bar{\Psi} = \Psi^\dagger\gamma^0 = \begin{pmatrix}\phi^\dagger & -\chi^\dagger\end{pmatrix}. \tag{2.72}$$

The Dirac Lagrangian written in terms of ϕ and χ with explicit powers of c is given by

$$\mathcal{L} = i\frac{\phi^\dagger\dot{\phi}}{c} + i\frac{\chi^\dagger\dot{\chi}}{c} + i\chi^\dagger\sigma^k\partial_k\phi + i\phi^\dagger\sigma^k\partial_k\chi. \tag{2.73}$$

Defining the conjugate momentum for each of the fields

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial(\dot{\phi})} = i\frac{\phi^\dagger}{c}, \quad \pi_\chi = \frac{\partial \mathcal{L}}{\partial(\dot{\chi})} = i\frac{\chi^\dagger}{c}, \tag{2.74}$$

where $\frac{\partial \mathcal{L}}{\partial(\dot{\phi}(\text{ or } \dot{\chi}))}$ stands for the right derivative of the field[i], one obtains the action in Hamiltonian form for a Dirac fermion field Ψ expressed in terms of its two-components spinors ϕ and χ

$$S\left[\phi, \chi, \pi_\phi, \pi_\chi\right] = \int dt \left[\int d^3x \, \left(\pi_\phi \dot{\phi} + \pi_\chi \dot{\chi}\right) - H\right], \tag{2.75}$$

$$H = \int d^3x \, \mathcal{E}, \quad \mathcal{E} = -c[\pi_\chi \sigma^k \partial_k \phi - \pi_\phi \sigma^k \partial_k \chi]. \tag{2.76}$$

With the direct $c \to 0$ limit, the Hamiltonian vanishes. However, in contrast with the previous section, there is a possible rescaling preserving the kinetic term $\int dt \int d^3x \, \left(\pi_\phi \dot{\phi} + \pi_\chi \dot{\chi}\right)$ which implies a non-vanishing Hamiltonian. Indeed, taking

$$\pi_\chi \to \frac{\pi_\chi}{c}, \quad \chi \to c\chi, \tag{2.77}$$

the rescaled Hamiltonian takes the form

$$H = \int d^3x \left[-\pi_\chi \sigma^k \partial_k \phi - c^2 \pi_\phi \sigma^k \partial_k \chi\right], \tag{2.78}$$

which implies the Carrollian action in the $c \to 0$ limit

$$S = \int dt d^3x \left[\pi_\phi \dot{\phi} + \pi_\chi \dot{\chi} + \pi_\chi \sigma^k \partial_k \phi\right]. \tag{2.79}$$

The equations of motion are given by

$$\dot{\phi} = 0 \quad \text{and } \dot{\chi} = -\sigma^k \partial_k \phi, \tag{2.80}$$

which are the same as those obtained with the Carrollian limit of the massless Dirac equation that we discussed in 2.2.2. The Carrollian theory described by the action (2.79) is Carroll invariant since the energy density $\mathcal{E}(x^k) = -\pi_\chi \sigma^k \partial_k \phi$ is a scalar under spatial translations and rotations and implies $\{\mathcal{E}(x), \mathcal{E}(x')\} = 0$, which are the two necessary and sufficient conditions to fulfilled for the Carroll invariance.

As discussed in section 2.1, there only exist two ways of producing a bosonic Carroll invariant theory: the magnetic theory is obtained by dropping the time derivatives (conjugate momenta) and keeping the spatial gradients in the energy density $\mathcal{E}$; the electric limit is obtained by doing the opposite, i.e., keeping the time derivatives and dropping the spatial gradients. In the Hamiltonian (2.78), we started with two spatial gradients (times a conjugate momenta) and the result was a Carrollian action with one of the spatial gradients dropped. By the definition of the electric contraction, one is led to identify this limit as a sort of magnetic one. The result that the general behavior of the electric and magnetic contractions that we discussed in section

[i] This is a standard convention in the Hamiltonian formulation of fermionic field theory (see the chapter on "Formal development of fermionic path integrals" of [28]).

2.1 is not retrieved is certainly a consequence of the fact that the Dirac action is of first-order in time derivative. This particularity implies that the Dirac Hamiltonian is of the same order in the time derivative than the Lagrangian.

In this chapter, we have therefore been able to use the analysis of our review of the literature to obtain a Carrollian limit of the massless Dirac action, which gives the same equations of motion as the ones found in [14]. We have identified that in the case of the Galilean and Carrollian limit, the decomposition into the two-components ϕ, and χ, of the Dirac field, as well as their rescaling, plays an important role. Applying this to the analysis of [11], we obtained a Carrollian limit with a Hamiltonian that is non-vanishing in the limit.

Chapter 3

Hamiltonian formulation of general relativity

As the basic dynamical variable of general relativity is the space-time metric, and as the most common way to derive Einstein's equations from an action principle is using an action in Lagrangian form, the standard formalism to introduce general relativity is the Lagrangian formalism. However, as in classical mechanics and, more generally, in field theory, there exists an equivalent description through the Hamiltonian formalism. Compared to other field theories, this transition to the canonical formalism is not so obvious. Indeed, in general relativity, time and space are treated equally, whereas in the Hamiltonian formalism, time and space have to be separated [29]. To reach this goal, the four-dimensional space-time manifold is foliated by spacelike hypersurfaces. With this foliation, one can decompose the spacetime into "space"+ "time" [30]. There exist several approaches to the Hamiltonian formulation of general relativity, but the one we develop in this chapter is the ADM formalism, from its authors Richard Arnowitt, Stanley Deser and Charles W. Misner. This formalism is useful to define an initial-value problem of general relativity, where one specifies some initial data on a hypersurface and then evolves these data in time. We will briefly comment on this in the following. In the frame of this thesis, the Hamiltonian formulation of general relativity will be used in chapter 4, dedicated to the study of the coupling of Carrollian gravity to spin-1/2 fermionic matter. Indeed, to achieve this, we will rewrite the Einstein-Cartan action coupled to massless Dirac fermions using the link between the first order and the Hamiltonian formulations of general relativity. In order to perform this rewriting, it is therefore first necessary to study the Hamiltonian formalism of general relativity.

As indicated previously, to realize the rewriting of general relativity in its Hamiltonian form, the foliation of space-time into spacelike hypersurfaces is required. We therefore start this chapter with some definitions and notions about hypersurfaces. Then, we undertake the study of the Hamiltonian (ADM) formulation of general relativity. We set $c = 1$ in this chapter, the powers of c will be reinstituted by dimensional analysis in the next chapter. This chapter follows the book of E. Poisson "A relativist's toolkit" [17]. To complete certain explanations, the book "3+1 Formalism in General Relativity" by E. Gourgoulhon [30] is also used.

3.1 Hypersurfaces

3.1.1 Definitions

A *hypersurface* Σ is a three-dimensional submanifold in a four-dimensional space-time that can be either timelike, spacelike, or null. It can be defined in two ways. A first way is to define a hypersurface as the set of points for which a scalar field Φ on the space-time manifold is constant. With the constant set to zero, this condition reads

$$\Phi(x^\alpha) = 0. \tag{3.1}$$

A particular hypersurface Σ is therefore selected by putting a restriction on the coordinates, $\Phi(x^\alpha) = 0$. The second way is to define a hypersurface by giving parametric equations of the form

$$x^\alpha = x^\alpha(x^i), \tag{3.2}$$

where x^i are coordinates intrinsic to the hypersurface. For example, a two-sphere is a hypersurface in a three-dimensional flat space that can be described by the parametric equations $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, and $z = R\cos\theta$, where θ and ϕ are intrinsic coordinates, or by the restriction on the coordinates $\Phi(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$, where R is the sphere's radius.

Let Σ be a hypersurface defined by the equation $\Phi(x^\alpha) = 0$. The vector $\partial_\alpha\Phi$ is normal to Σ and satisfies the following properties:

- $\partial_\alpha\Phi$ is timelike if Σ is spacelike;
- $\partial_\alpha\Phi$ is spacelike if Σ is timelike;
- $\partial_\alpha\Phi$ is null if Σ is null.

If the hypersurface is not null, we can re-normalize this vector to introduce a *unit normal* n_α satisfying

$$\mathrm{n}^\alpha \mathrm{n}_\alpha = \varepsilon \equiv \begin{cases} -1 & \text{if } \Sigma \text{ is spacelike} \\ +1 & \text{if } \Sigma \text{ is timelike} \end{cases}. \tag{3.3}$$

Given that the space-time is endowed with a metric tensor $g_{\alpha\beta}$, we would like now to define an induced metric h_{ij} on a hypersurface Σ. Using the parametric equations $x^\alpha(x^i)$, we can define tangent vectors contained in Σ as

$$e^\alpha_i = \frac{\partial x^\alpha}{\partial x^i}. \tag{3.4}$$

This implies in particular that $e^\alpha_i \mathrm{n}_\alpha = 0$. The metric induced to the hypersurface is obtained by restricting the line element ds^2 to displacements within the hypersurface. In practice, this means that

$$\begin{aligned} ds^2_\Sigma &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial x^i} dx^i\right) \left(\frac{\partial x^\beta}{\partial x^j} dx^j\right) \\ &= h_{ij} dx^i dx^j, \end{aligned} \tag{3.5}$$

where $h_{ij} = g_{\alpha\beta} e^\alpha_i e^\beta_j$ is the *induced metric* of the hypersurface. It transforms as a scalar under transformations of the space-time coordinates x^α, and as a tensor under transformations of the hypersurface coordinates. Such objects are called *three-tensors*.

Concerning the inverse metric $g^{\alpha\beta}$, it verifies the following relation in the non-null case

$$g^{\alpha\beta} = \varepsilon \mathrm{n}^\alpha \mathrm{n}^\beta + h^{ij} e^\alpha_i e^\beta_j, \tag{3.6}$$

where h^{ij} is the inverse of the induced metric. Equations such as (3.6) are called completeness relations.

3.1.2 Differentiation of tangent vector fields

On a given hypersurface Σ, one may encounter tensor fields $A^{\alpha\beta\cdots}$ that are purely tangent to this hypersurface. They can be decomposed in terms of basis vectors e^α_i on Σ:

$$A^{\alpha\beta\ldots} = A^{ij\cdots} e^\alpha_i e^\beta_j \ldots, \tag{3.7}$$

This three-tensor $A^{ij\cdots}$, associated with the tensor $A^{\alpha\beta\cdots}$, can be obtained with the projection

$$A_{\alpha\beta\ldots} e^\alpha_i e^\beta_j \cdots = A_{ij\ldots} \equiv h_{il} h_{jm} \cdots A^{lm\cdots}. \tag{3.8}$$

Now that we have defined tensor fields tangent to the hypersurface and their associated three-tensors, we will briefly describe how they are differentiated. If we restrict ourselves to the case of a tangent vector field A^α, the *intrinsic covariant derivative* of a three-vector A_i is defined to be the projection of the usual covariant derivative $\nabla_\beta A_\alpha$ onto the hypersurface:

$$\nabla_j A_i \equiv \nabla_\beta A_\alpha e^\alpha_i e^\beta_j. \tag{3.9}$$

The object $\nabla_j A_i$ defined here corresponds precisely to the covariant derivative of A_i, defined in the conventional manner in terms of a connection $\Gamma^i{}_{jl}$ that is compatible with the induced metric h_{ij},

$$\nabla_j A_i = \partial_j A_i - \Gamma_{ijl} A^l, \tag{3.10}$$

where

$$\Gamma_{ijl} = \frac{1}{2}\left(\partial_l h_{ij} + \partial_j h_{il} - \partial_i h_{jl}\right) \tag{3.11}$$

are the Christoffel symbols of the Levi-Civita connection for the induced metric on the hypersurface.

The three-tensor $\nabla_j A_i = \nabla_\beta A_\alpha e^\alpha_i e^\beta_j$ defined just above are the tangential components of the vector $\nabla_\beta A^\alpha e^\beta_j$. We would like now to investigate if this vector also possesses a normal component. To achieve that, $\nabla_\beta A^\alpha e^\beta_j$ is expressed in the form $g^\alpha_\mu \nabla_\beta A^\mu e^\beta_j$, and (3.6) is used. This provides

$$\begin{aligned}\nabla_\beta A^\alpha e^\beta_j &= \left(\varepsilon \mathrm{n}^\alpha \mathrm{n}_\mu + h^{im} e^\alpha_i e_{m\mu}\right) \nabla_\beta A^\mu e^\beta_j \\ &= \varepsilon \left(\mathrm{n}_\mu \nabla_\beta A^\mu e^\beta_j\right) \mathrm{n}^\alpha + h^{im} \left(\nabla_\beta A_\mu e^\mu_m e^\beta_j\right) e^\alpha_i.\end{aligned} \tag{3.12}$$

One can see that there is a term normal and a term tangent to the hypersurface. Using (3.9) and the fact that, as A^μ is tangent to the hypersurface, $n_\mu A^\mu = 0$, this reduces to

$$\nabla_\beta A^\alpha e^\beta_j = \nabla_j A^i e^\alpha_i - \varepsilon A^i \left(\nabla_\beta \mathrm{n}_\mu e^\mu_i e^\beta_j\right) \mathrm{n}^\alpha. \tag{3.13}$$

Defining the three-tensor

$$K_{ij} \equiv \nabla_\beta n_\alpha e^\alpha_i e^\beta_j, \tag{3.14}$$

one obtains,

$$\nabla_\beta A^\alpha e^\beta_j = \nabla_j A^i e^\alpha_i - \varepsilon A^i K_{ij} \mathrm{n}^\alpha. \tag{3.15}$$

Therefore, the normal component of the vector $\nabla_\beta A^\alpha e^\beta_j$ is given by the quantity $-\varepsilon A^i K_{ij} \mathrm{n}^\alpha$. The object K_{ij} is called the *extrinsic curvature*, or the *second fundamental form*, of the hypersurface Σ. This is an essential quantity in the Hamiltonian formulation of gravity, as we will discuss in the next section. The extrinsic curvature is a *symmetric* tensor

$$K_{ij} = K_{ji}, \tag{3.16}$$

and it can be rewritten as

$$K_{ij} = \frac{1}{2} \left(\mathcal{L}_n g_{\alpha\beta}\right) e^\alpha_i e^\beta_j, \tag{3.17}$$

where $\mathcal{L}_n g_{\alpha\beta}$ is the Lie derivative of the metric with respect to the normal vector n^α. The trace of the extrinsic curvature is given by

$$K \equiv h^{ij} K_{ij} = \nabla_\alpha n^\alpha. \tag{3.18}$$

From its definition in expression (3.14), one can see that the extrinsic curvature tells us how the normal vector n_α changes across the hypersurface Σ [31]. It is therefore said that K_{ij} is concerned with the *extrinsic* aspects of the hypersurface, i.e., the way in which the hypersurface is embedded in the space-time manifold.

Thus, we have that the induced metric h_{ij} is involved in the purely *intrinsic* part of the hypersurface's geometry, while K_{ij} is involved with the *extrinsic* aspects of the hypersurface. Together, these two objects offer an almost exhaustive characterization of the hypersurface.

3.1.3 Gauss-Codazzi equations

Using the covariant derivative (3.10), one can define a purely intrinsic curvature tensor

$$\nabla_i A^l_j - \nabla_j A^l_i = -R^l{}_{mij} A^m, \tag{3.19}$$

with

$$R^k{}_{lij} = \partial_i \Gamma^k{}_{lj} - \partial_j \Gamma^k{}_{li} + \Gamma^k{}_{mi} \Gamma^m{}_{lj} - \Gamma^k{}_{mj} \Gamma^m{}_{li}. \tag{3.20}$$

It can be proven that it can be expressed in terms of the four-dimensional Riemann tensor $\mathcal{R}^\gamma{}_{\delta\alpha\beta}$ as

$$\mathcal{R}_{\alpha\beta\gamma\delta} e^\alpha_i e^\beta_j e^\gamma_l e^\delta_m = R_{ijlm} + \varepsilon \left(K_{im} K_{jl} - K_{il} K_{jm}\right). \tag{3.21}$$

One can also prove that the full Riemann tensor can be rewritten in terms of the extrinsic curvature such as

$$\mathcal{R}_{\mu\alpha\beta\gamma} \mathrm{n}^\mu e^\alpha_i e^\beta_j e^\gamma_l = \nabla_l K_{ij} - \nabla_j K_{il}. \tag{3.22}$$

The details of these proofs can be found in [17]. The equations (3.21) and (3.22) are named *Gauss-Codazzi* equations. They relate the intrinsic and extrinsic properties of the hypersurface with the geometric properties of the space-time in which the hypersurface is embedded [32]. Moreover, these equations can be written in a contracted form using the Einstein tensor $\mathcal{G}_{\alpha\beta} = \mathcal{R}_{\alpha\beta} - \frac{1}{2}\mathcal{R}g_{\alpha\beta}$, with $\mathcal{R}_{\alpha\beta} = g^{\mu\nu}\mathcal{R}_{\mu\alpha\nu\beta}$ and $\mathcal{R} = g^{\alpha\beta}\mathcal{R}_{\alpha\beta}$, as well as the completeness relation (3.6):

$$-2\varepsilon\mathcal{G}_{\alpha\beta}\mathrm{n}^\alpha\mathrm{n}^\beta = {}^3R + \varepsilon\left(K^{ij}K_{ij} - K^2\right), \tag{3.23}$$

$$\mathcal{G}_{\alpha\beta}e^\alpha_i\mathrm{n}^\beta = \nabla_j K^j_i - \partial_i K, \tag{3.24}$$

where ${}^3R = h^{ij}R^m{}_{imj}$ is the three-dimensional Ricci scalar[a]. The equations (3.23) and (3.24) form part of the Einstein field equations on a hypersurface Σ, and play an important role in the *initial-value problem* of general relativity that we will briefly discuss in the following.

Initial-value problem

In the context of classical mechanics, the initial conditions on the position and velocity of a moving body are required to obtain a complete solution of the equations of motion. In field theories, this statement is extended to the requirement of the specification of the field and its time derivative at one instant of time. As in general relativity the field is $g_{\alpha\beta}$, it would be expected that a complete solution requires the specification of $g_{\alpha\beta}$ and $\partial_t g_{\alpha\beta}$ at one instant of time. This noncovariant statement can be converted into a more geometrical one, defining the *initial value problem* of general relativity.

The initial value problem starts by selecting a spacelike hypersurface Σ which represents an "instant of time". The initial values for the space-time metric are then given by the six components of the induced metric $h_{ij} = g_{\alpha\beta}e^\alpha_i e^\beta_j$. Thus, there are four arbitrary components of the metric, and this reflects the fact that we have complete freedom in the choice of coordinates x^α in general relativity. The extrinsic curvature carries information about the derivative of the metric in the direction normal to the spacelike hypersurfaces, which corresponds to the timelike direction. This is evident based on its expression $K_{ij} = \frac{1}{2}\mathcal{L}_n\left(g_{\alpha\beta}\right)e^\alpha_i e^\beta_j$. The extrinsic curvature is therefore a relevant choice for the initial values of the "time derivative" of the metric. In the context of general relativity, the initial-value problem therefore consists in the specification of h_{ij} and K_{ij} on a spacelike hypersurface Σ. These initial values, however, cannot be chosen freely as they have to satisfy the *constraints equations* of general relativity, given by (3.23), and (3.24) together with the Einstein field equations $\mathcal{G}_{\alpha\beta} = 8\pi\mathcal{T}_{\alpha\beta}$, with $\mathcal{T}_{\alpha\beta}$, the energy-momentum tensor,

$${}^3R + K^2 - K^{ij}K_{ij} = 16\pi\mathcal{T}_{\alpha\beta}\mathrm{n}^\alpha\mathrm{n}^\beta \equiv 16\pi\rho, \tag{3.25}$$

and

$$\nabla_j K^j_i - \partial_i K = 8\pi\mathcal{T}_{\alpha\beta}e^\alpha_i\mathrm{n}^\beta \equiv 8\pi j_i, \tag{3.26}$$

where the fact that $\varepsilon = -1$ for a spacelike hypersurface has been used. In the next section, dedicated to the Hamiltonian formulation of general relativity, we will see that the remaining components of the Einstein field equations give the evolution equations for h_{ij} and K_{ij}.

[a] As the Ricci scalar has no indices to indicate that this is a quantity defined on the hypersurface, we insist with this notation that it is a three-dimensional quantity.

The $3+1$ decomposition of space-time that will be presented in the next section allows formulating the problem of the resolution of the Einstein equations as an initial-value problem (or *Cauchy problem*) with constraints. As a matter of fact, it is worth mentioning that this is the basis for numerical relativity. Although, in this thesis, the use of this decomposition will be restricted to the rewriting of general relativity in the Hamiltonian formulation. More information about the initial value problem in general relativity can be found, e.g., in [30].

3.2 Hamiltonian formulation of general relativity

3.2.1 3+1 decomposition

Consider an arbitrary region $\mathscr{V}$ of the space-time manifold, bounded by a closed hypersurface $\partial\mathscr{V}$. The action functional of general relativity is given by the Einstein-Hilbert action,

$$S_G[g] = \frac{1}{16\pi G_N}\int_{\mathscr{V}} d^4x\sqrt{-g}\mathcal{R}, \tag{3.27}$$

where $\mathcal{R}$ is the Ricci scalar in $\mathscr{V}$. For simplicity, and because they will not be needed for the subsequent content of this thesis, we ignore any boundary terms in the gravitational action. The boundary terms that accompanied the Einstein-Hilbert action in the gravitational action and their role are briefly discussed in the appendix B.

As said in the introduction, to express the gravitational action in Hamiltonian form, it is necessary to perform a decomposition of space-time into "space"+ "time". This is realized by foliating $\mathscr{V}$ with a family of spacelike hypersurfaces, one for each "instant of time".

As explained in the previous section dedicated to hypersurfaces, one can define a hypersurface by putting a restriction on the coordinates. Here, the description of more than one hypersurface is needed. This can be done by introducing a scalar field $t(x^\alpha)$, such that $t(x^\alpha) = C$, with C a constant, describes a family of non-intersecting spacelike hypersurfaces Σ_t foliating $\mathscr{V}$. Considering our family of hypersurfaces, one can isolate for instance a single hypersurface by setting $C = 0$, and introducing a coordinate system $x^\alpha = (t, x, y, z)$ of $\mathscr{V}$ such that $t \in \mathbb{R}$, and (x, y, z) are Cartesian coordinates. The hypersurface Σ is then defined by the coordinate condition $t = 0$, with the coordinates $x^i = (x, y, z)$ on this particular hypersurface [30]. This "time function" only needs to satisfy two conditions: firstly, that t be a single-valued function of x^α, and secondly, that the unit normal $\mathrm{n}^\alpha \propto \partial_\alpha t$ to the hypersurface be a future-directed timelike vector field. On each hypersurface, one introduces a coordinate system x^i. Even if the coordinates between the hypersurfaces are not necessarily linked with one another, it is convenient to introduce a relationship between the coordinates on each hypersurface for the foliation. This relation is represented in figure 3.1. Consider a set of non-intersecting curves γ (also called a congruence of curves) intersecting the hypersurfaces Σ_t. It is not assumed that these curves intersect the hypersurfaces orthogonally. We use t as parameter along the curves, such that the vector t^α is tangent to the congruence. With this construction, one has the relation

$$t^\alpha\partial_\alpha t = 1. \tag{3.28}$$

By observing the figure 3.1, it becomes apparent that the curves γ establish connections between various points across different hypersurfaces, such as P, P', and P'' in the figure. This

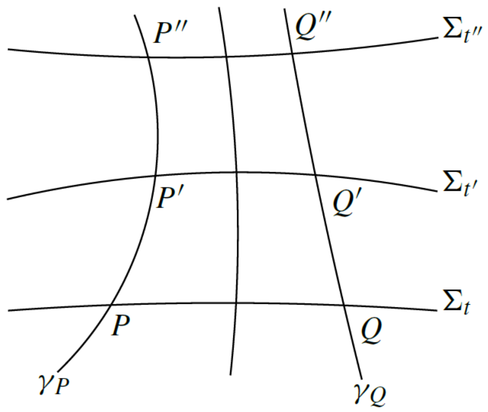

Figure 3.1: Foliation of space-time by spacelike hypersurface [17].

mapping between points on each hypersurface through the curves γ can be utilized to construct a coordinate system that is well-suited for the foliation. This can be done by fixing the coordinates of P' and P'', given $x^i(P)$ on Σ_t, by imposing $x^i(P'') = x^i(P') = x^i(P)$. Therefore, x^i remains constant along each curve γ, and this defines a coordinate system (t, x^i) in $\mathscr{V}$. In the following, we will always refer to these coordinates as the *"ADM coordinates"*.

The transformations between the usual coordinates x^α and the ADM coordinates give us the tangent vector to the curve[b],

$$t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{x^i}, \tag{3.29}$$

and the tangent vectors on the hypersurface Σ_t

$$e_i^\alpha = \left(\frac{\partial x^\alpha}{\partial x^i}\right)_t. \tag{3.30}$$

In particular, these relations imply that, in the coordinates (t, x^i), the two vectors reduce to $t^\alpha = \delta_t^\alpha$ and $e_i^\alpha = \delta_i^\alpha$. By definition of the tangent vectors to the hypersurface, we also have that

$$\mathcal{L}_t e_a^\alpha = 0. \tag{3.31}$$

Let us introduce the unit normal to the hypersurface

$$\mathrm{n}_\alpha = -N\partial_\alpha t, \tag{3.32}$$

with N, the scalar function called the *lapse*. The unit normal, with $\epsilon = -1$ for spacelike hypersurfaces, obeys

$$\mathrm{n}_\alpha \mathrm{n}^\alpha = -1, \quad \mathrm{n}_\alpha e_i^\alpha = 0. \tag{3.33}$$

One more object needs to be introduced in this $3+1$ decomposition of space-time: the three vector N^i known as the *shift*. It can be introduced via the expression of t^α in the basis provided by e_i^α and n^α,

$$t^\alpha = N\mathrm{n}^\alpha + N^i e_i^\alpha. \tag{3.34}$$

[b] The notation $t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{x^i}$ means the derivative of x^α with respect to t at x^i constant.

The figure 3.2 represents the situation. Let us now express the metric $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$

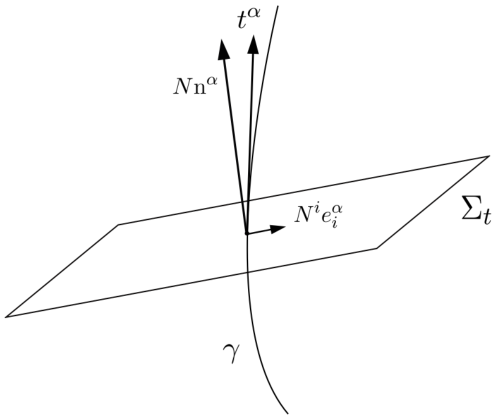

Figure 3.2: Decomposition of t^α into lapse and shift (inspired by [17]).

in terms of the ADM coordinates (t, x^i). We start by writing the change of coordinates for dx^α:

$$dx^\alpha = t^\alpha dt + e_i^\alpha dx^i \tag{3.35}$$

$$= (N dt)n^\alpha + (dx^i + N^i dt)e_i^\alpha, \tag{3.36}$$

where the expression of t^α (3.34) has been used, as well as the definition of the tangent vectors (3.29), and (3.30). Using (3.33), the line element $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ is then given by

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{3.37}$$

where $h_{ij} = g_{\alpha\beta}e_i^\alpha e_j^\beta$ is the induced metric on Σ_t. We therefore get the following expression for the metric components in the coordinates (t, x^i):

$$g_{\alpha\beta} = \begin{pmatrix} N^l N_l - N^2 & N_i \\ N_j & h_{ij} \end{pmatrix}. \tag{3.38}$$

We would like now to express the metric determinant in terms of $h \equiv \det[h_{ij}]$. The component of the metric g^{tt} is given by $g^{tt} = \text{cofactor}(g_{tt})/g = h/g$, but also by the change of coordinates $g^{tt} = g^{\alpha\beta}\partial_\alpha t \partial_\beta t = g^{\alpha\beta}\mathrm{n}_\alpha \mathrm{n}_\beta N^{-2} = -N^{-2}$, where expressions (3.32) and (3.33) were used. Combining the two, one obtains

$$\sqrt{-g} = N\sqrt{h}. \tag{3.39}$$

The equations (3.34), (3.37), and (3.39) are commonly recognized as the fundamental results of the $3+1$ decomposition.

3.2.2 Field theory

Let us pursue with the description of the Hamiltonian formulation of a field theory in this "$3+1$ decomposition formalism". As in [17], we consider here a scalar field φ for simplicity. This can easily be applied to any tensorial type of field. In this formalism, the vector t^α defined as (3.34) can be seen as the *time flow* vector which generates the diffeomorphisms that map Σ_{t_0} into Σ_{t_0+t} such as (3.28) is satisfied [32]. Given two hypersurfaces of the foliation, the time evolution of the fields can be conceptualized as the manner in which these fields change between the two hypersurfaces. With this in mind, the "time derivative" of a field φ is defined as the *Lie derivative* along the vector t^α

$$\dot{\varphi} \equiv \mathcal{L}_t \varphi. \tag{3.40}$$

In the ADM coordinates (t, x^i), one has $t^\alpha = \delta^\alpha_t$, and the Lie derivative reduces to the usual time derivative

$$\mathcal{L}_t \varphi = \frac{\partial \varphi}{\partial t}. \tag{3.41}$$

This is true for any tensor field using the ADM coordinate system adapted to the foliation. Using this definition, the *field's canonical momentum* π is defined by

$$\pi = \frac{\partial \left(\sqrt{-g}\mathcal{L}\right)}{\partial \dot{\varphi}}, \tag{3.42}$$

so that in the ADM coordinates, it reduces to its usual definition. Then, one can also define the *spatial derivative* of the field

$$\partial_i \varphi \equiv \partial_\alpha \varphi e^\alpha_i. \tag{3.43}$$

Finally, the Hamiltonian density is obtained via the Legendre transformation

$$\mathcal{H}\left(\pi, \varphi, \partial_i \varphi\right) = \pi \dot{\varphi} - \sqrt{-g}\mathcal{L}, \tag{3.44}$$

and the Hamiltonian functional is defined as

$$H[\pi, \varphi] = \int_{\Sigma_t} d^3x \; \mathcal{H}(\pi, \varphi, \partial_i \varphi). \tag{3.45}$$

With this Hamiltonian, the action functional is given by

$$S = \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \; \left(\pi \dot{\varphi} - \mathcal{H}\right). \tag{3.46}$$

We consider here, and in the following of this section, a region $\mathscr{V}$ of the manifold, foliated by spacelike hypersurfaces Σ_t bounded by closed two-surfaces S_t. The region $\mathscr{V}$ is itself delimited by two spacelike hypersurfaces Σ_{t_1} and Σ_{t_2} as well as a timelike hypersurface $\mathscr{B}$ (see figure 3.3). The Hamilton form of the fields equations is carried out in [17] by varying the action with respect to φ and π, such that $\delta\varphi$ vanish on the boundaries Σ_{t_1}, Σ_{t_2}, and $\mathscr{B}$. The final result is

$$\dot{\pi} = -\frac{\partial \mathcal{H}}{\partial \varphi} + \partial_i \left(\frac{\partial \mathcal{H}}{\partial \left(\partial_i \varphi\right)}\right), \quad \dot{\varphi} = \frac{\partial \mathcal{H}}{\partial \pi}, \tag{3.47}$$

which are the Hamilton's equation for a scalar field φ and its conjugate momentum π.

3.2.3 Gravitational action

Having established the $3+1$ formalism, we can now apply it to the Gravitational action

$$S_G = \frac{1}{16\pi G_N} \int_{\mathscr{V}} d^4x \; \sqrt{-g}\mathcal{R}, \tag{3.48}$$

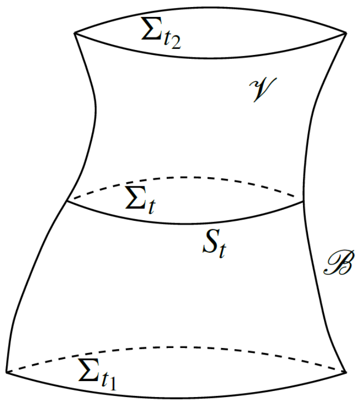

Figure 3.3: The region $\mathscr{V}$, its boundary $\partial\mathscr{V}$, and their foliations [17].

with the same set-up as above. In the following, n^α, x^i, h_{ij}, and K_{ij} will refer to the spacelike hypersurfaces Σ_t.

It has been established through a rigorous proof, detailed in section 3.5.3 of [17], that the Ricci scalar evaluated on the hypersurface Σ can be expressed in terms of the extrinsic curvature as

$$\mathcal{R} = {}^3R + K^{ij}K_{ij} - K^2 - 2\nabla_\alpha\left(\nabla_\beta \mathrm{n}^\alpha \mathrm{n}^\beta - \mathrm{n}^\alpha \nabla_\beta \mathrm{n}^\beta\right), \tag{3.49}$$

where ${}^3R$ is the Ricci scalar constructed from the induce metric h_{ij}. Using the fundamental results of the $3+1$ decomposition (3.39), written as $d^4x\ \sqrt{-g} = dtd^3x\ N\sqrt{h}$, we have that

$$\begin{aligned}\int_{\mathscr{V}} d^4x\sqrt{-g}\mathcal{R} = &\int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3xN\sqrt{h}\left({}^3R + K^{ij}K_{ij} - K^2\right) \\ &- 2\int_{\mathscr{V}} d^3xN\sqrt{h}\nabla_\alpha\left(\nabla_\beta \mathrm{n}^\alpha \mathrm{n}^\beta - \mathrm{n}^\alpha\nabla_\beta \mathrm{n}^\beta\right).\end{aligned} \tag{3.50}$$

The last term gives a boundary term using the Gauss-Stokes theorem, so we can disregard it in the action. The gravitational action in the $3+1$ decomposition gives thus

$$S_G = \frac{1}{16\pi G_N}\int_{t_1}^{t_2} \mathrm{d}t\left\{\int_{\Sigma_t} d^3xN\sqrt{h}\left({}^3R + K^{ij}K_{ij} - K^2\right)\right\}. \tag{3.51}$$

3.2.4 Gravitational Hamiltonian

The action (3.51) is to be considered as a functional of the variables h_{ij}, N, N^i (which describe the full space-time metric components $g_{\alpha\beta}$, c.f. (3.37)), and their time derivatives $\dot{h}_{ij}$, $\dot{N}$, $\dot{N}^i$ [30]. As a first step towards the Hamiltonian formulation, let us define the conjugate momentum to the metric h_{ij}. To accomplish that, following the definition of the time derivative of the field (3.40), S_G must be expressed in terms of

$$\dot{h}_{ij} \equiv \mathcal{L}_t h_{ij}, \tag{3.52}$$

where t^α is the vector field defined by equation (3.34). Using the definition of the induced metric, we have,

$$\dot{h}_{ij} = \mathcal{L}_t\left(g_{\alpha\beta}e_i^\alpha e_j^\beta\right) = \left(\mathcal{L}_t g_{\alpha\beta}\right)e_i^\alpha e_j^\beta, \tag{3.53}$$

where equation (3.31) is used. The Lie derivative of the metric is then given by

$$\begin{aligned}\mathcal{L}_t g_{\alpha\beta} &= \nabla_\beta t_\alpha + \nabla_\alpha t_\beta \\ &= \nabla_\beta \left(N \mathrm{n}_\alpha + N_\alpha\right) + \nabla_\alpha \left(N \mathrm{n}_\beta + N_\beta\right) \\ &= \mathrm{n}_\alpha \partial_\beta N + \partial_\alpha N \mathrm{n}_\beta + N \left(\nabla_\beta \mathrm{n}_\alpha + \nabla_\alpha \mathrm{n}_\beta\right) + \nabla_\beta N_\alpha + \nabla_\alpha N_\beta,\end{aligned} \tag{3.54}$$

with $N^\alpha = N^i e_i^\alpha$, and using equation (3.34). In the end, the time derivative of the metric is given by

$$\dot{h}_{ij} = 2N \left(K_{ij} - \nabla_j N_i - \nabla_i N_j\right), \tag{3.55}$$

where the definitions of the extrinsic curvature and the intrinsic covariant differentiation that we defined in section 3.1 are used. One can therefore express the extrinsic curvature K_{ij} in terms of the time derivative of the metric

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - \nabla_j N_i - \nabla_i N_j\right). \tag{3.56}$$

The dependence in $\dot{h}_{ij}$ of the gravitational action is therefore realized through the extrinsic curvature. One may also notice that there are no time derivatives of N and N^i in the Lagrangian, meaning that the conjugate momenta to N and N^i are not defined. This also indicates that the lapse function N and the shift N^i are not dynamical variables, which is in line with the fact that these two objects only serve to specify the foliation of $\mathscr{V}$. Indeed, the foliation being arbitrary, we have complete freedom in the choice of the lapse function and the shift vector. Based on its usual definition, the conjugate momentum is given by

$$\pi^{ij} = \frac{\partial\left(\sqrt{-g}\mathcal{L}_G\right)}{\partial \dot{h}_{ij}}. \tag{3.57}$$

As the dependence in $\dot{h}_{ij}$ of the gravitational action is realized through the extrinsic curvature, the momentum can be rewritten as

$$\left(16\pi G_N\right)\pi^{ij} = \frac{\partial K_{mn}}{\partial \dot{h}_{ij}} \frac{\partial}{\partial K_{mn}}\left(\left(16\pi G_N\right)\sqrt{-g}\mathcal{L}_G\right), \tag{3.58}$$

where

$$\left(16\pi G_N\right)\sqrt{-g}\mathcal{L}_G = \left[{}^3R + \left(h^{im}h^{jl} - h^{ij}h^{ml}\right)K_{ij}K_{ml}\right]N\sqrt{h}. \tag{3.59}$$

Evaluating the latter expression explicitly, provides

$$\left(16\pi G_N\right)\pi^{ij} = \sqrt{h}\left(K^{ij} - K h^{ij}\right). \tag{3.60}$$

Then, using the Legendre transformation $\mathcal{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L}$, as well as equations (3.55), (3.58), and the Lagrangian (3.51), one can develop

$$\begin{aligned}(16\pi G_N)\mathcal{H}_G &= \sqrt{h}\left(K^{ij} - K h^{ij}\right)\left(2NK_{ij} + \nabla_j N_i + \nabla_i N_j\right) - \left({}^3R + K^{ij}K_{ij} - K^2\right)N\sqrt{h} \\ &= \sqrt{h}N\left(K^{ij}K_{ij} - K^2 - {}^3R\right) + 2\left(K^{ij} - Kh^{ij}\right)\nabla_i N_j \sqrt{h} \\ &= \nabla_j\left(K^{ij}K_{ij} - K^2 - {}^3R\right)N\sqrt{h} - 2\nabla_j\left(K^{ij} - Kh^{ij}\right)N_i\sqrt{h}.\end{aligned} \tag{3.61}$$

The second line is obtained using the definition of the trace of the extrinsic curvature $K = h^{ij}K_{ij}$ and the fact that it is symmetric. The last line is obtained with an integration by parts in the last term. Finally, the gravitational Hamiltonian action is given by

$$S_G = \int dt \int_{\Sigma_t} d^3x \left(\pi^{ij}\dot{h}_{ij} - N\mathcal{H}_\perp - N^i\mathcal{H}_i\right) \tag{3.62}$$

with

$$\mathcal{H}_\perp = \frac{\sqrt{h}}{16\pi G_N}\left(K^{ij}K_{ij} - K^2 - {}^3R\right), \qquad \mathcal{H}^i = -\frac{2\sqrt{h}}{16\pi G_N}\nabla_j\left(K^{ij} - Kh^{ij}\right) \tag{3.63}$$

and

$$\pi^{ij} = -\frac{\sqrt{h}}{16\pi G_N}\left(K^{ij} - Kh^{ij}\right). \tag{3.64}$$

In these expressions, K_{ij} stands for the function of h_{ij} and π^{ij} given explicitly as

$$K^{ij} = \frac{16\pi G_N}{\sqrt{h}}\left(\pi^{ij} - \frac{1}{2}\pi h^{ij}\right), \tag{3.65}$$

with $\pi = h_{ij}\pi^{ij}$.

3.2.5 Variation of the Hamiltonian

Now that the Hamiltonian of general relativity has been defined, let us focus onto the equations of motion. Hamilton's equations of motion are obtained by varying the action with respect to the variables h_{ij}, N, N^i and π^{ij}. First, to carry out the variation with respect to h_{ij} and π^{ij}, one has to express the Hamiltonian part of (3.62) in terms of these variables instead of K_{ij}:

$$S_G = \int dt \int_{\Sigma_t} d^3x \left(\pi^{ij}\dot{h}_{ij} - N\mathcal{H}_\perp - N^i\mathcal{H}_i\right), \tag{3.66}$$

$$\mathcal{H}_\perp = \frac{16\pi G_N}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - \frac{\sqrt{h}}{16\pi G_N}{}^3R, \qquad \mathcal{H}_i = -2\nabla_j\pi_i{}^j. \tag{3.67}$$

This action is varied with respect to N, N^i, h_{ij} and π^{ij}, all treated as independent variables, and with the variation restricted by the conditions

$$\delta N = \delta N^i = \delta h_{ij} = 0 \text{ on } S_t. \tag{3.68}$$

After a long calculation detailed in [17], the complete variation of the gravitational action is given by

$$\delta S_G = \int_{\Sigma_t}\left((\dot{h}_{ij} - \mathcal{H}_{ij})\delta\pi_{ij} - \left(\dot{\pi}^{ij} + \mathcal{P}^{ij}\right)\delta h_{ij} + \mathcal{H}_\perp\delta N + \mathcal{H}_i\delta N^i\right)d^3x, \tag{3.69}$$

$$\begin{aligned}\mathcal{P}^{ij} =& Nh^{1/2}G^{ij} - \frac{1}{2}Nh^{-1/2}\left(\pi^{lm}\pi_{lm} - \frac{1}{2}\pi^2\right)h^{ij} + 2Nh^{-1/2}\left(\pi^i_l\pi^{jl} - \frac{1}{2}\pi\pi^{ij}\right)\\ &- h^{1/2}\left(\nabla^iN^j - h^{ij}\nabla^lN_l\right) - h^{1/2}\nabla_l\left(h^{-1/2}\pi^{ij}N^l\right) + 2\pi^{l(i}\nabla_lN^{j)},\end{aligned} \tag{3.70}$$

with $G^{ij} = R^{ij} - \frac{1}{2}{}^3Rh^{ij}$, the three-dimensional Einstein tensor, and

$$\mathcal{H}_{ij} = 2Nh^{-1/2}16\pi\left(\pi_{ij} - \frac{1}{2}\pi h_{ij}\right) + 2\nabla_{(j}N_{i)}. \tag{3.71}$$

Requiring the action to be stationary provides the vacuum Einstein field equations in Hamiltonian form

$$\dot{h}_{ij} = \mathcal{H}_{ij},\ \dot{\pi}^{ij} = -\mathcal{P}^{ij},\ \mathcal{H}_{\perp} = 0,\ \mathcal{H}_i = 0. \tag{3.72}$$

The first two equations give the time evolution of h_{ij} and π^{ij}. The last two equations are the Hamiltonian constraints ($\mathcal{H}_{\perp} = 0$) and the momentum constraint ($\mathcal{H}_i = 0$) of general relativity. We already obtained them in section 3.1.3, where they were referred to as the constraints equations of general relativity. Now that we have derived all the Einstein's equations, we give more details about the initial value problem in the following.

With the Hamiltonian formulation of general relativity, one therefore has the time evolution equations of h_{ij} and K_{ij} to define the initial value problem. The strategy to solve the Einstein's equations is thus the following. The first step is to choose a certain foliation of space-time, with the specification of N and N^i as functions of the ADM coordinates. As explained in 3.1.3, there is complete freedom in the choice of coordinates, which implies that one can choose the lapse and the shift freely as they are associated with the choice of coordinates (t, x^i). One must then choose initial data for h_{ij} and K_{ij} such that the constraints equations

$${}^3R + K^2 - K^{ij}K_{ij} = 0, \quad \nabla_j\left(K^{ij} - Kh^{ij}\right) = 0 \tag{3.73}$$

are satisfied. Finally, using the equations $\dot{h}_{ij} = \mathcal{H}_{ij}, \dot{\pi}^{ij} = -\mathcal{P}^{ij}$, the initial values can be evolved in time. The equations (3.72) usually serve as the starting point for numerical relativity.

In this section, we have first rewritten the gravitational action in Hamiltonian form with a decomposition of space-time into "space" + "time". Then, we have investigated the equations of motion. We have also seen that this formalism is especially useful in the definition of the initial-value problem of general relativity. In the frame of this thesis, we will use the Hamiltonian formulation, as well as the "3+1" decomposition described in this section to study the Carrollian limit of Einstein's theory of gravity coupled to Dirac fermions.

Chapter 4

Coupling Carrollian gravity to fermionic matter

In section 2.1, we have followed the analysis conducted by [11], which explored the Carrollian limits of Lorentz-invariant theories through the Hamiltonian formalism. Among these theories, the Carroll limits of Einstein's theory of gravity were also studied. Through this analysis, two distinct limits of this theory were identified, namely, the magnetic and electric limits of Einstein's theory of gravity. The interest of this section is in the magnetic limit, which can be derived as follows. One starts from the Hamiltonian action that we have studied in section 3.2,

$$S_G = \int dt \int_{\Sigma_t} d^3x \left(\pi^{ij} \dot{h}_{ij} - N\mathcal{H}_\perp - N^i \mathcal{H}_i \right), \tag{4.1}$$

with

$$\mathcal{H}_\perp = \frac{16\pi G_M c^2}{\sqrt{h}} \left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2 \right) - \frac{\sqrt{h}}{16\pi G_M} {}^3R, \qquad \mathcal{H}_i = -2\nabla_j \pi_i{}^j, \tag{4.2}$$

where the explicit powers of the speed of light c are written, and where $G_M = c^{-4} G_N$, with G_N Newton's constant. As we have explained in section 2.1, the "direct" $c \to 0$ limit in the action leads to the magnetic theory, while the electric theory is obtained via a $c-$dependent rescaling of the fields. In [11], the identification

$$\mathcal{H}_M = -\frac{\sqrt{h}}{16\pi G_M} {}^3R, \quad \mathcal{H}_E = \frac{16\pi G_M c^2}{\sqrt{h}} \left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2 \right), \tag{4.3}$$

is performed, such as in the direct limit, the electric Hamiltonian $\mathcal{H}_E$ vanishes. Carrollian theories of gravity have also been constructed by gauging the Carroll algebra, for example in [16]. It was suspected that the magnetic action found through the Hamiltonian formulation of [11], and the action obtained in [16] through a gauging of the Carroll algebra, were equivalent. The aim of [15] was to prove this equivalence and clarify the links between these two actions.

In addition to the comparison of the two already discussed ultra-relativistic limits, the problem was also studied in the following way. Rather than directly examining the outcome of the Carrollian limit, the results have been revisited, starting from the relation between the first order and Hamiltonian formulations of gravity [33, 34, 35]. The aim of this part of the thesis is to pursue this work by adding the coupling to massless Dirac fermions to their analysis.

This chapter is organized as follows. Before getting to the heart of the matter, we will review some concepts necessary for the study of fermions in curved space: the vierbein formalism, the definition of a Lagrangian invariant under local Lorentz transformations and, finally, the first order formalism. Afterward, we will define the procedure of the "gauging" of an algebra. We will then rewrite the Einstein-Cartan action coupled with Dirac fermions in Hamiltonian form using the relation between the first order and the Hamiltonian formulation of general relativity. Finally, we will take the $c \to 0$ limit and analyze the results obtained.

4.1 General relativity with spinors

In this section, we review some concepts required to describe fermions coupled to gravity. First, we describe the *vierbein* formalism, that we will use throughout the following of this chapter. Second, we define the covariant derivative for fermions, in order to be able to define them in curved space-time with a Lagrangian invariant under local Lorentz transformations. Finally, we introduce the first order formalism of gravity coupled with fermions, which is an equivalent way to the usual second order formalism in which general relativity is defined. The content of the two first sections draws from the book "Geometry, Topology and Physics", by M. Nakahara [36], and the last section from the book "Supergravity" by D.Z. Freedman and A. Van Proeyen [37]. We set $c = 1$ in this section.

4.1.1 Non-coordinate bases

The metric is of central importance in general relativity. However, when dealing with fermions in curved space-time, it is convenient to use the *vierbein* formalism, introduced in the following paragraphs.

Let us consider a D-dimensional differentiable Lorentzian manifold M equipped with a metric g[a]. In coordinate basis, the tangent space T_pM and the cotangent space T_p^*M are spanned respectively by $\{\partial/\partial x^\mu\}$ and $\{dx^\mu\}$. Consider the change of basis

$$\hat{E}_A = E_A^\mu \frac{\partial}{\partial x^\mu}, \tag{4.4}$$

with E_A^μ a square $D \times D$ change of basis matrix $\in GL(D, \mathbb{R})$, with $\det(E_A^\mu) > 0$. We require $\hat{E}_A$ to be orthonormal with respect to g,

$$g(\hat{E}_A, \hat{E}_B) = g_{\mu\nu} E_A^\mu E_B^\nu \stackrel{!}{=} \eta_{AB}. \tag{4.5}$$

Then, we introduce the inverse of E_A^μ, E_μ^A, such that

$$E_A^\mu E_\nu^A = \delta_\nu^\mu, \quad E_B^\mu E_\mu^A = \delta_B^A. \tag{4.6}$$

This allows to introduce the dual basis $\hat{E}^A$

$$\hat{E}^A = E_\mu^A dx^\mu, \tag{4.7}$$

[a] M is said to be Lorentzian if the signature of the metric is $(-,+,+,+)$, which is the case in this thesis.

such that

$$\hat{E}_A(\hat{E}^B) = \delta^B_A. \tag{4.8}$$

One can easily inverse expression (4.5), giving,

$$g_{\mu\nu} = E^A_\mu E^B_\nu \eta_{AB}. \tag{4.9}$$

The bases $\hat{E}_A$ and $\hat{E}^A$ are called the non-coordinate bases. In the following of this thesis, we will use the capital Latin indices $A, B, \ldots$ to refer to these bases. In arbitrary dimension D, the coefficients E^A_μ are called the vielbeins, and in four-dimensions they are called the vierbeins.

The connection coefficients with respect to the coordinate basis can be defined by

$$\nabla_A \hat{E}_B \equiv \Omega_A{}^C{}_B \hat{E}_C = \Omega_A{}^C{}_B E^\mu_C \frac{\partial}{\partial x^\mu} \equiv \Omega_A{}^\mu{}_B \ \frac{\partial}{\partial x^\mu} \tag{4.10}$$

This relation can be expressed using the definition of the coordinate basis as

$$\nabla_A \hat{E}_B = E^\mu_A \left(\partial_\mu E^\nu_B + E^\lambda_B \Gamma^\nu{}_{\mu\lambda}\right) \frac{\partial}{\partial x^\nu} \equiv \Omega_A{}^C{}_B E^\mu_C \frac{\partial}{\partial x^\mu}, \tag{4.11}$$

and contracting with E^A_ρ, we have

$$\partial_\rho E^\nu_B + \Gamma^\nu{}_{\rho\lambda} E^\lambda_B - \Omega_\rho{}^C{}_B E^\nu_C = 0. \tag{4.12}$$

This last relation is known as the *vielbein postulate* (or vierbein postulate if we are in four-dimensions), and it can be rewritten as

$$\nabla_\mu E^\nu_A = 0. \tag{4.13}$$

One defines the connection one-form $\Omega^A{}_B \equiv \Omega_C{}^A{}_B \hat{E}^C$. This allows to introduce the torsion two-form, $\mathcal{T}^A \equiv \frac{1}{2} \mathcal{T}_{BC}{}^A \hat{E}^B \wedge \hat{E}^C$, as well as the curvature two-form $\mathcal{R}^A{}_B = \frac{1}{2} \mathcal{R}^A{}_{BCD} \hat{E}^C \wedge \hat{E}^D$, via *Cartan's structure equations*

$$d\hat{E}^A + \Omega^A{}_B \wedge \hat{E}^B = \mathcal{T}^A, \tag{4.14}$$

$$d\Omega^A{}_B + \Omega^A{}_C \wedge \Omega^C{}_B = \mathcal{R}^A{}_B. \tag{4.15}$$

Starting from the dual basis $\hat{E}^A$, one can always obtain another orthonormal basis $\hat{E}'^A$ by a *local* Lorentz transformation

$$\hat{E}'^A(p) = \Lambda^A{}_B(p) \hat{E}^B(p), \tag{4.16}$$

at each point p. The local Lorentz transformations are defined as the transformations preserving the flat local metric η_{AB}. The transformation rule of the spin connection can be found by requiring that the torsion $\mathcal{T}^A$ transforms as a vector:

$$\Omega'^A{}_B = \Lambda^A{}_C \Omega^C{}_D \left(\Lambda^{-1}\right)^D{}_B + \Lambda^A{}_C \left(d\Lambda^{-1}\right)^C{}_B. \tag{4.17}$$

Finally, let us discuss the Levi-Civita connection in a non-coordinate basis. Recall that the Levi-Civita connection is the unique connection on a manifold with metric g that is metric compatible and torsion-free. The metric compatibility condition takes the form

$$\Omega_{AB} = -\Omega_{BA}. \tag{4.18}$$

The torsion-free condition reads simply

$$d\hat{E}^A + \Omega^A{}_B \wedge \hat{E}^B = 0. \tag{4.19}$$

As we will elaborate in the next section, the non-coordinates bases introduced here are of great importance for defining spinors in curved space-time.

4.1.2 Spinors in curved space-time

To address gravity coupled with fermionic matter in the forthcoming sections, it is essential to begin by defining spinors that are coupled with a curved background. To define a field theory in the presence of gravity, one can use the minimal coupling procedure where one starts from the action in flat space and substitute the flat metric with an arbitrary metric, ordinary derivatives with covariant derivatives, and the integration measure d^4x with $d^4x\sqrt{-g}$. This can also be applied to the case of Dirac fermions. However, there is one peculiarity for fermionic theories with respect to other theories containing only bosonic fields. Spinors are defined by their transformation properties under special Lorentz transformations, meaning that local frames, or vierbeins, that we have introduced in the last section, are a necessity to treat spinors in general relativity, as they allow to define Lorentz transformations in each point. In the following, we start from the Dirac Lagrangian minimally coupled and define the covariant derivative such as the Lagrangian is invariant under local Lorentz transformations.

Let Ψ be a Dirac spinor in four dimensions and let γ^A denote the 4×4 complex Dirac matrices satisfying the Clifford algebra $\left\{\gamma^A, \gamma^B\right\} = -2\eta^{AB}$. A Dirac spinor transforms under a local Lorentz transformation $\Lambda^A{}_B(p)$ as

$$\Psi(p) \to \rho(\Lambda)\Psi(p), \quad \bar{\Psi} \to \bar{\Psi}(p)\rho(\Lambda)^{-1}, \tag{4.20}$$

where $\rho(\Lambda)$ is the spinor representation of Λ. For the Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\gamma^A\nabla_A - m)\Psi \tag{4.21}$$

to be invariant, one has to seek a covariant derivative which is a local Lorentz vector and transform as a spinor,

$$\nabla_A\Psi \to \rho(\Lambda)\Lambda_A{}^B\nabla_B\Psi. \tag{4.22}$$

Suppose that ∇_A has the form

$$\nabla_A\Psi = E^\mu_A\left[\partial_\mu + \Theta_\mu\right]\Psi, \tag{4.23}$$

and note that

$$E^\mu_A\partial_\mu\Psi \to \Lambda_A{}^B E^\mu_B\partial_\mu\rho(\Lambda)\Psi = \Lambda_A{}^B E^\mu_B\left[\rho(\Lambda)\partial_\mu\Psi + \partial_\mu\rho(\Lambda)\Psi\right]. \tag{4.24}$$

From (4.24) and (4.22), one has that Θ_μ must satisfy

$$\Theta_\mu \to \rho(\Lambda)\Theta_\mu\rho(\Lambda)^{-1} - \partial_\mu\rho(\Lambda)\rho(\Lambda)^{-1}. \tag{4.25}$$

We would like now to find the explicit form of Θ_μ such as it fulfills the transformation (4.25). To achieve that, we consider an infinitesimal Lorentz transformation $\Lambda_A{}^B = \delta_A{}^B + \varepsilon_A{}^B$, where ε_{AB} is antisymmetric. Under this transformation, the Dirac spinor obeys

$$\Psi \to \exp\left[\frac{1}{2}\varepsilon^{AB}\Sigma_{AB}\right]\Psi \simeq \left(1+\frac{1}{2}\varepsilon^{AB}\Sigma_{AB}\right)\Psi, \tag{4.26}$$

where $\Sigma_{AB} \equiv -\frac{1}{4}\left[\gamma_A, \gamma_B\right]$ is the spinor representation of the generators of the Lorentz algebra

$$\left[\Sigma_{AB}, \Sigma_{CD}\right] = \eta_{AC}\Sigma_{DB} + \eta_{BD}\Sigma_{CA} - \eta_{AD}\Sigma_{CB} - \eta_{BC}\Sigma_{DA}. \tag{4.27}$$

With this Lorentz transformation, (4.25) becomes

$$\begin{aligned}\Theta_\mu \to &\left(1+\frac{1}{2}\varepsilon^{AB}\Sigma_{AB}\right)\Theta_\mu\left(1-\frac{1}{2}\varepsilon^{CD}\Sigma_{CD}\right) - \frac{1}{2}\partial_\mu\varepsilon^{AB}\Sigma_{AB}\left(1-\frac{1}{2}\varepsilon^{CD}\Sigma_{CD}\right)\\ &= \Theta_\mu + \frac{1}{2}\varepsilon^{AB}\left[\Sigma_{AB}, \Theta_\mu\right] - \frac{1}{2}\partial_\mu\varepsilon^{AB}\Sigma_{AB}.\end{aligned} \tag{4.28}$$

If we define

$$\Theta_\mu \equiv \frac{1}{2}\Omega_\mu{}^{AB}\Sigma_{AB}, \tag{4.29}$$

knowing that the infinitesimal version of the local Lorentz transformations of the components of the spin connection (c.f. eq. (4.17)),

$$\Omega_\mu{}^{AB} \to \Omega_\mu{}^{AB} + \varepsilon^A{}_C\Omega_\mu{}^{CB} - \Omega_\mu{}^A{}_C\varepsilon^{CB} - \partial_\mu\varepsilon^{AB}, \tag{4.30}$$

the transformation (4.25) is satisfied. Indeed, one has

$$\begin{aligned}\frac{1}{2}\Omega_\mu{}^{AB}\Sigma_{AB} \to &\frac{1}{2}\left(\Omega_\mu{}^{AB} + \varepsilon^A{}_C\Omega_\mu{}^{CB} - \Omega_\mu{}^A{}_C\varepsilon^{CB} - \partial_\mu\varepsilon^{AB}\right)\Sigma_{AB}\\ &= \frac{1}{2}\Omega_\mu{}^{AB}\Sigma_{AB} + \frac{1}{2}\varepsilon^{AB}\left[\Sigma_{AB}, \frac{1}{2}\Omega_\mu{}^{CD}\Sigma_{CD}\right] - \frac{1}{2}\partial_\mu\varepsilon^{AB}\Sigma_{AB},\end{aligned} \tag{4.31}$$

where the last line is obtained using the Lorentz algebra (4.27). Finally, the following expression for the covariant derivative is obtained,

$$\nabla_\mu\Psi = \partial_\mu\Psi + \frac{1}{2}\Omega_\mu{}^{AB}\Sigma_{AB}\Psi, \tag{4.32}$$

with $\Sigma_{AB} = -\frac{1}{4}\left[\gamma_A, \gamma_B\right]$. The Lagrangian

$$\mathcal{L} = \bar{\Psi}\left[i\gamma^A E^\mu{}_A\left(\partial_\mu + \frac{1}{2}\Omega_\mu{}^{AB}\Sigma_{AB}\right) - m\right]\Psi \tag{4.33}$$

is therefore a scalar under coordinate and local Lorentz transformations. In the subsequent section, we will utilize the definition of the covariant derivative derived in this section. Specifically, we will examine the coupling between spinors and gravity in the first order formalism, where the vierbein and the spin-connection are treated as two independent fields.

4.1.3 First order formalism for gravity and fermions

General relativity is usually introduced in the second order formalism, in which the metric, or the vierbein, is the dynamical variable. In that case, the covariant derivative and curvature tensors are constructed from the torsion-free connections $\Gamma^{\rho}{}_{\mu\nu}(g)$ and $\Omega_{\mu}{}^{AB}(e)$. There exists however another way to describe gravity, which is through the first order formalism, where one starts with an action in which the vierbein and the spin connection are independent variables. In this formalism, the assumption that the connection is free of torsion is not made; this will be derived as an equation of motion. Without matter, the equations of motion obtained by the variation of $\Omega_{\mu}{}^{AB}$ set the spin connection in terms of the vierbein, $\Omega_{\mu}{}^{AB} = \Omega_{\mu}{}^{AB}(e)$ as in the second order formalism. It is therefore straightforward to switch between the two formulations. However, if one considers the coupling of gravity with fermionic matter, the equations of motion with respect to $\Omega_{\mu}{}^{AB}$ give a contributing term to the torsion. This contribution implies that the $\Omega_{\mu}{}^{AB}$ field equation gives the relation $\Omega_{\mu}{}^{AB} = \Omega_{\mu}{}^{AB}(e) + \mathcal{K}_{\mu}{}^{AB}$, where $\mathcal{K}_{\mu}{}^{AB}$, which is the contorsion tensor defined in terms of the torsion tensor $\mathcal{T}_{\mu\nu\rho} = g_{\rho\sigma}\mathcal{T}_{\mu\nu}{}^{\sigma}$ as

$$\mathcal{K}_{\mu[\upsilon\rho]} = -\frac{1}{2}\left(\mathcal{T}_{[\mu\nu]\rho} - \mathcal{T}_{[\upsilon\rho]\mu} + \mathcal{T}_{[\rho\mu]\nu}\right), \tag{4.34}$$

is determined as a quadratic expression in the spinor fields. It can be proven that substituting this into the first order action will lead to terms in the action quartic in the spinor fields. We will observe this result in section 4.4, where we will explicitly re-express the spin connection using the equations of motion.

The Einstein-Hilbert action of general relativity expressed in terms of the metric $g_{\mu\nu}$, that we have used in the description of the Hamiltonian formulation of general relativity, can be expressed in terms of the vierbein E^A_μ. This gives the second-order action where the Ricci scalar is only a function of the vierbein. As explained at the beginning of this section, this is not the only choice. We can indeed consider the first order action

$$S_G[E^A_\mu, \Omega_\mu{}^{AB}] = \frac{1}{16\pi G_N}\int d^4xE\ E^\mu_A E^\nu_B \mathcal{R}_{\mu\nu}{}^{AB}[\Omega], \tag{4.35}$$

with the Riemann tensor expressed in terms of the spin-connection $\Omega_\mu{}^{AB}$,

$$\mathcal{R}_{\mu\nu}{}^{AB} = 2\partial_{[\mu}\Omega_{\nu]}{}^{AB} + 2\Omega_{[\mu}{}^{AC}\Omega_{\nu]}{}^{DB}\eta_{CD}, \tag{4.36}$$

and the relation $E^2 = (\det(E^\mu_A))^2 = -g$ between the determinant of the vierbein and the one of the metric. This action is the *Einstein-Cartan* action, where the spin-connection $\Omega_\mu{}^{AB}$ and the vierbein E^A_μ appear as independent variables. In this section, we would like to describe the coupling of general relativity to fermionic Dirac fields in this first order formalism that was just described. In order to achieve that, we use the following massless Dirac action in curved space,

$$S_{1/2} = \int d^4xE\Big(\frac{i}{2}\bar{\Psi}\gamma^A E^\mu_A \nabla_\mu\Psi - \frac{i}{2}\bar{\Psi}\overleftarrow{\nabla}_\mu\gamma^A E^\mu_A\Psi\Big), \tag{4.37}$$

with

$$\nabla_\mu\Psi = (\partial_\mu - \frac{1}{8}\Omega_\mu{}^{AB}[\gamma_A,\gamma_B])\Psi \quad \text{and} \quad \bar{\Psi}\overleftarrow{\nabla}_\mu = \bar{\Psi}(\overleftarrow{\partial}_\mu + \frac{1}{8}\Omega_\mu{}^{AB}[\gamma_A,\gamma_B]). \tag{4.38}$$

This action results from the addition of a covariant derivative with respect to the one with the

Lagrangian that we have defined in the section concerning spinors in curved space-time (c.f. (4.33)). In the first order formalism, as we have said above and that we will demonstrate in the following, the coupling to fermions implies that the connection is not torsion-free. However, a term proportional to the torsion appears when integrating by parts in that case (see appendix C for the proof of this result). The actions (4.33) and (4.37) are therefore inequivalent in the first order formalism. In the literature that we follow in this thesis to carry out the coupling of fermions to Carrollian gravity, the action (4.37) is used. We will therefore use this convention to make easier the comparison of our results with those in the literature later on.

Let us now carry out the variation of the coupled action $S_{1/2} + S_G$ with respect to the spin connection $\Omega_\mu{}^{AB}$. Defining $\kappa^2 = 8\pi G_N$ to streamline the calculations, one can find that the variation of the gravitational action is given by

$$\delta S_G = \frac{1}{2\kappa^2} \int d^4x \; E \; E^\mu_A E^\nu_B \left(\nabla_\mu \delta\Omega_\nu{}^{AB} - \nabla_\nu \delta\Omega_\mu{}^{AB} + \mathcal{T}_{\mu\nu}{}^\rho \delta\Omega_\rho{}^{AB} \right), \tag{4.39}$$

which reduces to

$$\delta S_G = \frac{1}{2\kappa^2} \int d^4x \; E \; \left(2\nabla_\mu \left(E^\mu_A E^\nu_B \delta\Omega_\nu{}^{AB} \right) + E^\mu_A E^\nu_B \mathcal{T}_{\mu\nu}{}^\rho \delta\Omega_\rho{}^{AB} \right), \tag{4.40}$$

using the A, B anti-symmetry of the spin connection and the vierbein postulate given by (4.13). Performing an integration by parts of the first term, we arrive to

$$\delta S_G = \frac{1}{2\kappa^2} \int d^4x \; E \; \Big(2 \; \mathcal{T}_{\rho A}{}^\rho E^\nu_B + \mathcal{T}_{AB}{}^\nu \Big) \delta\Omega_\nu{}^{AB}, \tag{4.41}$$

or if we make the A, B anti-symmetry explicit,

$$\delta S_G = \frac{1}{2\kappa^2} \int d^4x \; E \; \Big(\; \mathcal{T}_{\rho A}{}^\rho E^\nu_B - \; \mathcal{T}_{\rho B}{}^\rho E^\nu_A + \mathcal{T}_{AB}{}^\nu \Big) \delta\Omega_\nu{}^{AB}. \tag{4.42}$$

Let us now perform the variation of the massless Dirac action (4.37) with respect to the spin connection. Introducing the completely antisymmetric object $\gamma_{\mu AB} \equiv \frac{1}{4}\{\gamma_\mu, [\gamma_A, \gamma_B]\}$, the variation of the massless Dirac action in this antisymmetric form is given by

$$\delta S_{1/2} = \int \; d^4x \; E \; \Big(-\frac{i}{16} \bar{\Psi} \gamma^\mu \delta\Omega_\mu{}^{AB} [\gamma_A, \gamma_B] \Psi - \frac{i}{16} \bar{\Psi} \delta\Omega_\mu{}^{AB} [\gamma_A, \gamma_B]) \gamma^\mu \Psi \Big) \tag{4.43}$$

$$= -\frac{i}{4} \int d^4x \; E \; \bar{\Psi} \gamma^\mu{}_{AB} \Psi \; \delta\Omega_\mu{}^{AB}. \tag{4.44}$$

If one combines now (4.41), and (4.44), such as $\delta S_G + \delta S_{1/2} = 0$, one arrives to

$$\frac{1}{2\kappa^2} \; \Big(\; \mathcal{T}_{\rho A}{}^\rho E^\nu_B - \; \mathcal{T}_{\rho B}{}^\rho E^\nu_A + \mathcal{T}_{AB}{}^\nu \Big) = \frac{i}{4} \bar{\Psi} \gamma^\nu{}_{AB} \Psi. \tag{4.45}$$

By contracting this equation with η^{AB}, one can see that the trace of the last torsion in the left-hand side vanishes, as the trace of the right-hand side is zero due to the anti-symmetry of

${\gamma^{\nu}}_{AB}$. Finally, the expression of the torsion in the presence of spin-1/2 Dirac fields[b] is given by

$$\mathcal{T}_{AB}{}^{\nu} = \frac{i\kappa^2}{2}\bar{\Psi}{\gamma^{\nu}}_{AB}\Psi. \tag{4.46}$$

This expression will be derived in section (4.4) in the context of Carrollian gravity coupled to spin-1/2 fermionic matter.

4.2 General relativity as a gauge theory

A gauge theory is a theory enjoying a local symmetry, i.e. the transformations parameters of the symmetry depend on the space-time position. The action of the gauge theory is invariant under these transformations. Having this in mind, for the comprehension of this part of the thesis, we now need to define what it is meant by "gauging" an algebra in [15] and [16]. In this thesis, we restrict ourselves to an explanation based on an analogy with Yang-Mills theory. Readers interested in delving deeper into the concept of the gauging of an algebra are referred to the dedicated literature covering this topic (see, e.g., [38, 39]).

The gauging procedure proposed in the latter two papers can be compared to the case of a Yang-Mills theory. In this theory, a gauge field $A_\mu(x)$ is expanded in terms of the generator matrices T^a of a given compact group, such as $A_\mu(x) = A^a_\mu(x)T_a$, and the Lagrangian of the theory is invariant under gauge transformations of this field $A_\mu(x)$. The idea of the gauging procedure for the case of general relativity is to describe this theory as a gauge theory for the Poincaré group. The crucial difference between the general relativity and the Yang-Mills cases is that, for the case of general relativity, the action is not invariant under the whole Poincaré transformations of the gauge fields, but only under the one generated by its homogeneous subgroup, the Lorentz group. The connection components associated with translations are called in this approach the "solderings" forms (the vierbeins), and they are regarded as tangent vectors to the manifold. This endows the tangent spaces to the manifold with a structure on which the Lorentz (the homogeneous Poincaré) group acts. The starting point is the Poincaré algebra

$$\begin{aligned}
[M_{AB}, M_{CD}] &= \eta_{AC}M_{DB} + \eta_{BD}M_{CA} - \eta_{AD}M_{CB} - \eta_{BC}M_{DA}, \\
[M_{AB}, P_C] &= \eta_{CB}P_A - \eta_{CA}P_B, \\
[P_A, P_B] &= 0.
\end{aligned} \tag{4.47}$$

A connection one-form taking values on the Poincaré algebra can be defined as

$$A_\mu = E^A_\mu P_A + \frac{1}{2}{\Omega_\mu}^{AB}M_{AB}. \tag{4.48}$$

This step is the same as what is done for Yang-Mills theory. However, one can use the feature of the Poincaré algebra that some of its generators are translations to identify a basis of tangent (co)vectors to the manifold, as explained above. The gauge field E^A_μ is therefore the vierbein field, while ${\Omega_\mu}^{AB}$ is the spin-connection field. Under local gauge transformations, the Poincaré connection (4.48) transforms as $\delta A_\mu = \mathcal{D}_\mu\Gamma$ (Yang-Mills type transformation), where $\mathcal{D}_\mu =$

[b] Note here that if we had performed the variation with the action defined by the Lagrangian (4.33), one would have obtained the expression $T_{AB}{}^{\nu} = \frac{i\kappa^2}{2}\bar{\Psi}\gamma^{\nu}\gamma_{AB}\Psi$, showing that the two actions are indeed inequivalents.

$\partial_\mu \Gamma + [A_\mu, \Gamma]$ is the covariant derivative of the gauge parameter

$$\Gamma = \eta^A P_A + \frac{1}{2}\Theta^{AB}\Omega_{AB}. \tag{4.49}$$

In components, this gives the gauge field transformations:

$$\begin{aligned} \delta E_\mu^A &= \partial_\mu \eta^A + \Omega_\mu{}^{AB}\eta_B - E_\mu^B \Theta_B^A, \\ \delta \Omega_\mu{}^{AB} &= \partial_\mu \Theta^{AB} + \Omega_\mu{}^{C[A}\Theta^{B]C}. \end{aligned} \tag{4.50}$$

In the same way that the connection $A_\mu(x)$ is expanded in terms of the generators in Yang-Mills theory, so is the Field strength $F_{\mu\nu} = F^a_{\mu\nu} T^a$. This object is defined as the commutator of the covariant derivative with respect to the connection $A_\mu = A^a_\mu T^a$. In the context of general relativity, one can also define such an object, known as the *curvature* of the Poincaré connection

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = \mathcal{T}_{\mu\nu}{}^A P_A + \frac{1}{2}\mathcal{R}_{\mu\nu}{}^{AB} M_{AB}, \tag{4.51}$$

where the torsion tensor $\mathcal{T}_{\mu\nu}{}^A$ is defined as

$$\mathcal{T}_{\mu\nu}{}^A = 2\left(\partial_{[\mu} E^A_{\nu]} + \Omega_{[\mu}{}^{AB} E_{\nu]B}\right), \tag{4.52}$$

and the curvature tensor $\mathcal{R}_{\mu\nu}{}^{AB}$ as

$$\mathcal{R}_{\mu\nu}{}^{AB} = 2\left(\partial_{[\mu}\Omega_{\nu]}{}^{AB} + \Omega_{[\mu}^{\ \ AC}\Omega_{\nu]}^{\ \ DB}\eta_{CD}\right). \tag{4.53}$$

We consider the following action, known as the Einstein-Cartan action, which is invariant under local Lorentz transformations and general coordinate transformations:

$$S = \frac{1}{16\pi G_N}\int d^4x\ E\ E^\mu_A E^\nu_B \mathcal{R}_{\mu\nu}{}^{AB}, \tag{4.54}$$

with $E = \det(E^A_\mu)$, and where we have defined the inverse vierbein E^μ_A such as

$$E^A_\mu E^\mu_B = \delta^A_B, \quad E^A_\mu E^\nu_A = \delta^\nu_\mu. \tag{4.55}$$

Now that the case of general relativity is realized, let us move on to the gauging of the Carroll algebra.

4.3 Carrollian gravity as a gauge theory

The correspondence with Yang-Mills theory to define general relativity as a gauge theory can also be applied to the Carroll algebra to define Carroll gravity. The Carroll algebra (without the vanishing commutators of the space-time translations) takes the form

$$\begin{aligned} &[M_{ab}, M_{cd}] = \delta_{ac}M_{db} + \delta_{bd}M_{ca} - \delta_{ad}M_{cb} - \delta_{bc}M_{da}, \\ &[M_{ab}, B_d] = \delta_{bd}B_a - \delta_{ad}B_b, \quad [M_{ab}, P_c] = \delta_{cb}P_a - \delta_{ca}P_b, \\ &[M_{ab}, H] = 0, \qquad\qquad\qquad\quad [B_a, H] = 0, \\ &[B_a, P_b] = \delta_{ba}H, \qquad\qquad\quad\ \ [B_a, B_b] = 0, \end{aligned} \tag{4.56}$$

where we have decomposed the $A-$indices into $A = \{0, a\}$, with $a, b, \dots$ taking spatial values $1, 2, 3$. As explained in the subsection 1.4.2, one can obtain this algebra by a contraction of the Poincaré algebra, where the speed of light c goes to zero. However, one can also introduce $c = \epsilon \hat{c}$, with ϵ a dimensionless parameter, so that the limit corresponds to sending $\epsilon \to 0$. We set then $\hat{c} = 1$. This is how it is defined in [16][c], with the same rescaling of the Poincaré generators as introduced in the subsection 1.4.2,

$$\epsilon P_0 \equiv H, \tag{4.57}$$
$$\epsilon M_{0a} \equiv B_a, \tag{4.58}$$

but with c replaced by ϵ. The Carroll algebra defined above can be rewritten in a compact manner as

$$\begin{aligned} [M_{AB}, P_C] &= \zeta_{CB} P_A - \zeta_{CA} P_B, \\ [M_{AB}, M_{CD}] &= \zeta_{AC} M_{DB} + \zeta_{BD} M_{CA} - \zeta_{AD} M_{CB} - \zeta_{BC} M_{DA}, \end{aligned} \tag{4.59}$$

with the tensor ζ_{AB}, which is the degenerate Carroll metric that we have defined in section 1.4.1,

$$(\zeta_{AB}) = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ab} \end{pmatrix}, \tag{4.60}$$

and the vector

$$\left(n^B\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \zeta_{AB} n^B = 0. \tag{4.61}$$

As in the case of general relativity, the first step in the “gauging” of a Lie algebra is to define a connection one-form taking value on that algebra, here the Carrollian one

$$A_\mu = \tau_\mu H + e^a_\mu P_a + \omega^a_\mu B_a + \frac{1}{2} \omega_\mu{}^{ab} M_{ab}. \tag{4.62}$$

We can apply the same reasoning that we have used for general relativity to the Carrollian case here since some of the generators of the Carroll algebra are translations, and can be therefore identified with a basis of tangent (co)-vectors to the manifold. We have that $\left\{\tau_\mu, e^a_\mu\right\}$ constitute a basis of the cotangent space. In [15], the dual basis of the tangent space is represented as $\{n^\mu, e^\mu_a\}$, and we will maintain this notation. With the gauge parameter

$$\Gamma = \xi H + \xi^a P_a + \lambda^a B_a + \frac{1}{2} \lambda^{ab} M_{ab}, \tag{4.63}$$

the gauge fields transform as

$$\begin{aligned} \delta e^a_\mu &= \partial_\mu \xi^a + \omega_\mu{}^{ab} \xi_b - e^b_\mu \lambda^a_b, \\ \delta \tau_\mu &= \partial_\mu \xi + \omega^a_\mu \xi_a - e^a_\mu \lambda_a, \\ \delta \omega_\mu{}^{ab} &= \partial_\mu \lambda^{ab} + 2 \omega_\mu{}^{c[a} \lambda^{b]}_c, \\ \delta \omega^a_\mu &= \partial_\mu \lambda^a + \omega_\mu{}^{ab} \lambda_b - \omega^b_\mu \lambda^a_b. \end{aligned} \tag{4.64}$$

In the continuity of the definition of a gauging of general relativity in the previous section, we would like now to define an action invariant under the homogeneous Carroll subgroup, which

[c] To be more specific, they define it using a parameter ω tending to infinity. This is entirely equivalent to our approach here.

is the group obtained by contraction of the Lorentz group. In [16], this was derived by taking the ultra-relativistic limit of the Einstein-Cartan action. To define this limit, the gauge fields and symmetry parameters of general relativity are redefined with the same parameter ϵ used to obtain the Carroll contraction. The requirement was that the connection (4.62) and the gauge parameter (4.63) are invariant under the redefinitions of the generators (4.58). In other words, if we start with the connection (4.48) and the gauge parameters (4.49), with the redefinition of the generators (4.58) and the fields, we want to retrieve the ones associated with the Carroll group in the limit $\epsilon \to 0$. This condition leads to the following redefinitions of the gauge fields and parameters

$$\begin{aligned} E_\mu^0 &= \epsilon\tau_\mu, & \Omega_\mu{}^{0a} &= \epsilon\omega_\mu^a \\ E_\mu^a &= e_\mu^a, & \Omega_\mu{}^{ab} &= \omega_\mu{}^{ab}, \\ \eta^0 &= \epsilon\xi, & \Theta^{0a} &= \epsilon\lambda^a, \\ \eta^a &= \xi^a, & \Theta^{ab} &= \lambda^{ab}. \end{aligned} \tag{4.65}$$

Performing these redefinitions in the transformations (4.50) associated with the Poincaré group and taking the limit $\epsilon \to 0$, one retrieves the Carroll transformation (4.64). Using this same rescaling in the Einstein-Cartan action, and taking the $\epsilon \to 0$ limit, a Carrollian action invariant under local homogeneous Carroll transformations was derived in [16].

Now that we have defined the procedure of the "gauging" of an algebra, we will move on in the following section to the rewriting of the Einstein-Cartan action coupled to massless spin-1/2 Dirac fields in Hamiltonian form.

4.4 Magnetic limit of gravity coupled to Dirac fermions

A particular feature of Carrollian geometry which will be important in this section is that, contrary to the Riemannian case, there is no unique torsion-free, metric compatible connection [39]. One has instead a connection determined up to the addition of a term proportional to an arbitrary symmetric tensor. This means in particular that if we are in the first order formalism of general relativity, and we want to express the Carrollian spin connection in terms of the vierbein using the torsion constraints, one component of the connection will remain arbitrary. In the analysis performed in [15] to prove the equivalence between the Carrollian actions of [11] and [16], this component turns out to be proportional to the conjugate momenta to the spatial metric, and this allowed to recover the magnetic limit of the Hamiltonian formulation of gravity.

As explained at the beginning of this section, this equivalence between the two actions has also been derived from the relation between the first order and Hamiltonian formulation of gravity. To go from one to the other, as we will see in details, one needs to express all components of the spin connection but one in terms of the vierbein. In the end, if we are in the relativistic case, one can choose to express this latter connection in terms of the vierbein if we want and retrieve the second order formulation of gravity. However, when performing the $c \to 0$ limit, this choice is not available anymore because the connection component that play the role of conjugate momentum are arbitrary, and one is forced to keep the component associated with the conjugate momentum.

In the following of this section, we will complete the analysis of the section "Magnetic limit of the Einstein-Cartan action" of [15] by incorporating the coupling to massless Dirac fermions.

We will then take the analysis even further by using the rescaling of the Dirac fields that we introduced in the study of the Carrollian limit of the free Dirac action (c.f. section 2.2.3) to obtain a different limit.

4.4.1 Rewriting in 3+1 form

We start from the Einstein-Cartan action (4.35)

$$S_G = \frac{c^3}{16\pi G_N}\int dtd^3x\ E\ E^\mu_A E^\nu_B \mathcal{R}_{\mu\nu}{}^{AB}, \tag{4.66}$$

with the powers of c explicitly written. In this action, $\mathcal{R}_{\mu\nu}{}^{AB}$ stands for the Riemann tensor

$$\mathcal{R}_{\mu\nu}{}^{AB} = 2\partial_{[\mu}\Omega_{\nu]}{}^{AB} + 2\Omega_{[\mu}{}^{AC}\Omega_{\nu]}{}^{DB}\eta_{CD}, \tag{4.67}$$

with $\eta_{CD} = \mathrm{diag}\,(-1,+1,+1,+1)$, the Minkowski metric. In what follows, like in section 4.1.3, we define $\kappa^2 = 8\pi G_N$ to streamline the calculations. As in the previous section, we use the substitution $c = \epsilon\hat{c}$, where ϵ is a dimensionless parameter, so the limit corresponds to taking $\epsilon \to 0$, and we set c to one. Then, we will consider the same scaling in ϵ for the components of the vierbein and of the spin connection as in [16] and that we have discussed in the previous section with (4.65),

$$E^A_\mu = \left(\epsilon\tau_\mu, e^a_\mu\right), \quad \Omega^{AB}_\mu = \left(\epsilon\omega^a_\mu, \omega_\mu{}^{ab}\right). \tag{4.68}$$

Rescaling Newton's constant via $G_N = \epsilon^4 G_M$, and taking the limit $\epsilon \to 0$ leads to the action of [16] that we mentioned in the previous section.

With the massless Dirac action coupled to a curved background

$$S_{1/2} = \int dtd^3x\ E\ \left(\frac{i}{2}\bar{\Psi}\gamma^A E^\mu_A \nabla_\mu\Psi - \frac{i}{2}\bar{\Psi}\overleftarrow{\nabla}_\mu\gamma^A E^\mu_A\Psi\right), \tag{4.69}$$

with

$$\nabla_\mu\Psi = (\partial_\mu - \frac{1}{8}\Omega^{AB}_\mu[\gamma_A,\gamma_B])\Psi \quad \text{and} \quad \bar{\Psi}\overleftarrow{\nabla}_\mu = \bar{\Psi}(\overleftarrow{\partial}_\mu + \frac{1}{8}\Omega^{AB}_\mu[\gamma_A,\gamma_B]), \tag{4.70}$$

the coupling of the Einstein-Cartan action with massless Dirac fermions takes the form,

$$S = \int dtd^3x\ E\ \left[\frac{c^3}{16\pi G_N}E^\mu_A E^\nu_B\mathcal{R}_{\mu\nu}{}^{AB} + \left(\frac{i}{2}\bar{\Psi}\gamma^A E^\mu_A\nabla_\mu\Psi - \frac{i}{2}\bar{\Psi}\overleftarrow{\nabla}_\mu\gamma^A E^\mu_A\Psi\right)\right]. \tag{4.71}$$

We would like now to rewrite this action in Hamiltonian form. For this purpose, we need to link the first order and ADM formulation of general relativity that we have presented in the subsection 4.1.3 and in chapter 3, respectively. To achieve this, we introduce the vector tangent e^A_i and the normal n_A to the spacelike hypersurface Σ_t, starting from the components of the rescaled vierbein E^A_μ of expression (4.68) and its inverse E^μ_A in the ADM coordinates $x^\mu = (t, x^i)$:

$$e^A_i \equiv E^A_i, \quad \mathrm{n}_A \equiv -\epsilon N E^t_A, \tag{4.72}$$

where N is the lapse function. We stress here that, at this stage, the covector n_A should not be confused with the vector n^A defined in equation (4.61). The variables e^A_i and n_A satisfy the relations

$$\eta^{AB}\mathrm{n}_A\mathrm{n}_B = -1, \quad e_i^A\mathrm{n}_A = 0, \tag{4.73}$$

where the Minkowski metric η_{AB} is used to raise and lower the capital Latin indices. These are precisely the relations (3.33) for a spacelike hypersurface that we have defined in the previous chapter and that define the foliation of spacetime into spacelike hypersurfaces. The rescaling of ϵ in (4.72) is chosen such that the expression $\eta^{AB}\mathrm{n}_A\mathrm{n}_B = -1$ remains well-defined in the limit $\epsilon \to 0$. Using (4.72), all components of the vierbein and its inverse are determined in terms of e_i^A, n_A, the lapse N, and the shift N^i:

$$E_A^\mu = \left(-\epsilon^{-1}N^{-1}\mathrm{n}_A, e_A^i + \epsilon^{-1}N^{-1}N^i\mathrm{n}_A\right), \quad E_\mu^A = (\epsilon\ N\ \mathrm{n}^A + e_i^AN^i, e_i^A), \tag{4.74}$$

where the object e_A^i satisfies

$$e_A^ie_j^A = \delta_j^i, \quad e_A^ie_i^B = \delta_A^B + \mathrm{n}_A\mathrm{n}^B. \tag{4.75}$$

The second equation is the completeness relation, which was also introduced in the Hamiltonian formulation of general relativity. The latter quantity can also be defined as $e_A^i = h^{ij}e_j^B\eta_{AB}$, where h^{ij} is the inverse of the spatial metric[d]

$$h_{ij} = e_i^Ae_j^B\eta_{AB}. \tag{4.76}$$

The parametrization (4.74) of the vierbein and its inverse, that we can find for instance in [35], implies the usual ADM decomposition of the metric

$$g_{\mu\nu} = \begin{pmatrix} N^kN_k - \epsilon^2N^2 & N_i \\ N_i & h_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\epsilon^2N^2} & \frac{N^i}{\epsilon^2N^2} \\ \frac{N^i}{\epsilon^2N^2} & h^{ij} - \frac{N^iN^j}{\epsilon^2N^2} \end{pmatrix}. \tag{4.77}$$

This is exactly the same metric $g_{\mu\nu}$ as the one obtained in the Hamiltonian formulation of general relativity (see expression (3.37)), but without the ϵ rescaling. It is clear with (4.77) that the metric $g_{\mu\nu}$ is degenerate in the $\epsilon \to 0$ limit as the components of its inverse $g^{\mu\nu}$ becomes infinite in that case. In terms of the parametrization (4.74), the identity $\sqrt{-g} = N\sqrt{h}$ becomes $E = \epsilon N\sqrt{h}$. With this latter identity and the latter parametrization, the rewriting of the Einstein Cartan action was realized in [15],

$$S_G = \epsilon^3 \int \frac{dtd^3x\sqrt{h}}{16\pi G_N}\left[2\mathrm{n}_{[B}e^i_{A]}\mathcal{R}_{ti}{}^{AB} + \epsilon Ne^i_{[A}e^j_{B]}\mathcal{R}_{ij}{}^{AB} + 2N^{[i}\mathrm{n}_{[A}e^{j]}_{B]}\mathcal{R}_{ij}{}^{AB}\right]. \tag{4.78}$$

We have done the same with the massless Dirac action, which takes the form

$$\begin{aligned} S_{1/2} = \int dtd^3x\sqrt{h}\Big[& -\frac{i}{2}\bar{\Psi}\gamma^A\mathrm{n}_A\partial_t\Psi + \frac{i}{2}\bar{\Psi}\gamma^Ae_A^i\epsilon N\partial_i\Psi + \frac{i}{2}\bar{\Psi}\gamma^AN^i\mathrm{n}_A\partial_i\Psi \\ & + \frac{i}{2}\bar{\Psi}\overleftarrow{\partial}_t\gamma^A\mathrm{n}_A\Psi - \frac{i}{2}\bar{\Psi}\overleftarrow{\partial}_i\gamma^Ae_A^i\epsilon N\Psi - \frac{i}{2}\bar{\Psi}\overleftarrow{\partial}_i\gamma^AN^i\mathrm{n}_A\Psi \\ & + \frac{i}{4}\bar{\Psi}\left(\mathrm{n}_A\Omega_t{}^{CD} - e_A^i\epsilon N\Omega_i{}^{CD} - N^i\mathrm{n}_A\Omega_i{}^{CD}\right)\gamma^A{}_{CD}\Psi\Big]. \end{aligned} \tag{4.79}$$

[d]Note that the “spatial metric” is what we called the “induced metric” on the hypersurface in the previous chapter concerning the Hamiltonian formulation of general relativity. It will be named like this in this section to match [15], that we are following throughout this section.

Varying the complete action $S_G + S_{1/2}$ with respect to the full spin-connection $\Omega_\mu{}^{AB}$ provides the following contribution to the torsion:

$$\mathcal{T}_{\mu\nu}{}^C = \frac{i\kappa^2}{2\epsilon^3}\bar{\Psi}\gamma^C{}_{\mu\nu}\Psi, \tag{4.80}$$

with $\mathcal{T}_{\mu\nu}{}^C = 2\partial_{[\mu}E^C_{\nu]} + 2\Omega_{[\mu}{}^{CB}E^D_{\nu]}\eta_{BD}$. Without considering the coupling to fermionic matter, the equations of motion obtained from a variation with respect to the spin connection imposes the vanishing of the torsion. However, as discussed in section 4.1.3, the inclusion of fermions in the first order formalism of general relativity results in the introduction of a fermionic contribution to the torsion. We wish now to consider the variation of the action (4.79) with respect to the different components of the spin connection, and compare the results with those of [15]. The spin connection $\Omega_\mu{}^{AB}$ can be decomposed into its tangential and normal components to the hypersurface such as

$$\begin{aligned} \Omega_t{}^{ij} &= \Omega_t{}^{AB}e^i_A e^j_B, \quad \Omega_t{}^i{}_\perp = \Omega_t{}^{AB}e^i_A \mathrm{n}_B, \\ \Omega_i{}^{jk} &= \Omega_i{}^{AB}e^j_A e^k_B, \quad \Omega_{ij\perp} = \Omega_i{}^{AB}e_{jA}\mathrm{n}_B. \end{aligned} \tag{4.81}$$

The same can be done with the following relevant components of the torsion

$$\mathcal{T}_{ij\perp} = \mathcal{T}_{ij}{}^A\mathrm{n}_A = \frac{i\kappa^2}{2\epsilon^3}\bar{\Psi}\gamma^A{}_{BC}e^B_i e^C_j \mathrm{n}_A\Psi, \tag{4.82}$$

$$\mathcal{T}_{ij}{}^k = \mathcal{T}_{ij}{}^A e^k_A = \frac{i\kappa^2}{2\epsilon^3}\bar{\Psi}\gamma^A{}_{BC}e^B_i e^C_j e^k_A\Psi, \tag{4.83}$$

$$\mathcal{T}_{ti\perp} = \mathcal{T}_{ti}{}^A\mathrm{n}_A = \frac{i\kappa^2}{2\epsilon^3}\bar{\Psi}\gamma^A{}_{BC}e^B_l N^l e^C_i \mathrm{n}_A\Psi, \tag{4.84}$$

$$\mathcal{T}_{t[ij]} = \mathcal{T}_{t[i}{}^A e_{j]A} = \frac{i\kappa^2}{2\epsilon^2}\bar{\Psi}\gamma^A{}_{BC}N\mathrm{n}^B e^C_{[i}e_{j]A}\Psi + \frac{i\kappa^2}{2\epsilon^3}\bar{\Psi}\gamma^A{}_{BC}e^B_l N^l e^C_{[i}e_{j]A}\Psi. \tag{4.85}$$

Using the explicit expression of the torsion, these relations become[e]

$$2\partial_{[i}e^A_{j]}\mathrm{n}_A - 2\Omega_{[ij]\perp} = \frac{i\kappa^2}{2\epsilon^3}\bar{\Psi}\gamma^A{}_{BC}e^B_i e^C_j \mathrm{n}_A\Psi, \tag{4.86}$$

$$2\partial_{[i}e^A_{j]}e^k_A - 2\Omega_{[ij]}{}^k = \frac{i\kappa^2}{2\epsilon^3}\bar{\Psi}\gamma^A{}_{BC}e^B_i e^C_j e^k_A\Psi, \tag{4.87}$$

$$\begin{aligned} \partial_t e^A_i \mathrm{n}_A - N^j\partial_i e^A_j \mathrm{n}_A - \Omega_{ti\perp} + N^j\Omega_{ij\perp} + \epsilon\partial_i N \\ = \frac{i\kappa^2}{2\epsilon^3}\bar{\Psi}\gamma^A{}_{BC}e^B_l N^l e^C_i \mathrm{n}_A\Psi, \end{aligned} \tag{4.88}$$

$$\begin{aligned} \partial_t e^A_{[i}e_{j]A} + \epsilon N\partial_{[i}e^A_{j]}\mathrm{n}_A + e^A_{[i}\partial_{j]}N^k e_{kA} + N^k e^A_{[i}\partial_{j]}e_{kA} - \Omega_{tij} - \epsilon N\Omega_{[ij]\perp} - N^k\Omega_{[ij]k} \\ = \frac{i\kappa^2}{2\epsilon^2}\bar{\Psi}\gamma^A{}_{BC}N\mathrm{n}^B e^C_{[i}e_{j]A}\Psi + \frac{i\kappa^2}{2\epsilon^3}\bar{\Psi}\gamma^A{}_{BC}e^B_l N^l e^C_{[i}e_{j]A}\Psi. \end{aligned} \tag{4.89}$$

We may now compute the variation of the action (4.78), coupled to the action (4.79), with respect to the different spin connections. Firstly, varying with respect to $\Omega_t{}^{ij}$ and $\Omega_t{}^i{}_\perp$ gives respectively

$$\mathcal{T}_{ij\perp} = \frac{i\kappa^2}{2\epsilon^3}\bar{\Psi}\mathrm{n}_A\gamma^A{}_{CD}e^C_i e^D_j\Psi, \tag{4.90}$$

[e] There is a sign difference in the last expression compared to [15]. The correct expression is the one in this thesis.

and

$$\mathcal{T}_{ij}{}^{j} = \frac{-i\kappa^2}{2\epsilon^3}\bar{\Psi}\mathrm{n}_A\gamma^A{}_{CD}\mathrm{n}^C e_i^D\Psi. \tag{4.91}$$

Varying with respect to $\Omega_j{}^{ki}$ provides

$$\begin{aligned}\left(\delta_k^j\mathcal{T}_{ti\perp} - \delta_i^j\mathcal{T}_{tk\perp} + \delta_k^j\mathcal{T}_{il}{}^l\epsilon N - \delta_j^i\mathcal{T}_{kl}{}^l\epsilon N - N^l\delta_k^j\mathcal{T}_{li\perp} + N^l\delta_i^j\mathcal{T}_{lk\perp} - \mathcal{T}_{ik}{}^j\epsilon N - N^j\mathcal{T}_{ik\perp}\right)\\ = \frac{i\kappa^2}{2\epsilon^3}\bar{\Psi}\left(e_A^j\epsilon N + N^j\mathrm{n}_A\right)\gamma^A{}_{CD}e_k^C e_i^D\Psi.\end{aligned} \tag{4.92}$$

Finally, varying with respect to $\Omega_{[ij]\perp}$ gives

$$-2\mathcal{T}_{t[ij]} + N_i\mathcal{T}_{kj}{}^k - N_j\mathcal{T}_{ki}{}^k - 2N^l\mathcal{T}_{l[ji]} = \frac{i\kappa^2}{\epsilon^3}\bar{\Psi}\left(\epsilon N e_{[iA} + N_{[i}\mathrm{n}_A\right)\gamma^A{}_{CD}e_{j]}^D\mathrm{n}^C\Psi. \tag{4.93}$$

In the two last equations, there are two more terms than in [15] on the left-hand side. We were able to realize that the correct relationships are the ones above because the addition of fermions allows performing an additional cross-check. Indeed, by replacing the projections of the components of the torsion in the left-hand side of the above equations by their expressions (4.82)-(4.85), we were able to retrieve the same right-hand of these equations, allowing us to check the variation of the Einstein-Cartan action alone.

We have considered here the variation of the action with respect to all the components of the spin connection, except $\Omega_{(ij)\perp}$. This component is the one playing the role of the arbitrary symmetric tensor that we have talked about at the beginning of this section, and is related to the conjugate momenta of the spatial metric in the Hamiltonian formulation. In the Carrollian limit, one is forced to keep this component. We shall therefore not need the torsion constraint $\mathcal{T}_{t(ij)}$ obtained by varying the action with respect to $\Omega_{(ij)\perp}$.

Using the torsion constraint for $\mathcal{T}_{ij}{}^k$, taking a linear combination of the cyclic permutations of this constraint over the indices i, j and k, as described in [15], one obtains

$$e_A^k\partial_i e_j^A + \Omega_i{}^k{}_j - \Gamma^k{}_{ij} = -\frac{i\kappa^2}{4\epsilon^3}\bar{\Psi}\gamma_{BC}{}^A e_i^B e^{kC}e_{jA}\Psi, \tag{4.94}$$

where $\Gamma^k{}_{ij}$ are the Christoffel symbols of the Levi-Civita connection for the spatial metric that we already introduced in section 3.2 (c.f. (3.11)). We will refer to this equation in the following as the spatial vierbein postulate. There is an additional fermionic contribution compared to the one obtained in [15].

4.4.2 Rewriting of the action in Hamiltonian form

As we have seen in section 3.2.2 about the definition of field theory in Hamiltonian form, the time derivative appearing in the definition of the conjugate momentum is defined as the Lie derivative along the vector t^α of the field (c.f. (3.40)). In the ADM coordinates, t^α reduces to δ^α_t, and the Lie derivatives of e^A_i and Ψ reduce to the usual time derivative. We therefore define the conjugate momenta with respect to the vierbein e^A_i and the spinor field as

$$p^i_A \equiv \frac{\partial(\sqrt{h}\mathcal{L})}{\partial \dot{e}^A_i}, \quad \pi_\Psi \equiv \frac{\partial(\sqrt{h}\mathcal{L})}{\partial \dot{\Psi}}, \tag{4.95}$$

with $\dot{e}^A_i = \partial_t e^A_i$ and $\dot{\Psi} = \partial_t \Psi$. These definitions are in line with those of [33], which addresses the Hamiltonian formulation of the Einstein-Cartan action with massive Dirac fermions.

Let us recall that the theory presented here admits a Lorentz gauge symmetry, and, so far, we have been working without fixing any gauge. However, when studying the Hamiltonian formulation with vierbeins, it is common to impose the so-called "time gauge", because this greatly simplifies the calculations. In the treatment that we have given in this section, the theory is invariant under local Lorentz transformations. The time gauge consists in restricting this invariance. The most common condition to impose this gauge is to take E^0_μ to coincide at each point with the unit normal to the hypersurface [33]. With this condition, the theory is no longer invariant under the full local Lorentz group $SO(1,3)$, but only under the subgroup $SO(3)$ of spatial rotations which leave the normal invariant. The "time gauge" condition may be stated as $\mathrm{n}^A = \delta^A_0$. This implies that from now on, $\mathrm{n}^A = n^A = \delta^A_0$, and the normal covector is equal to the covector n^A defined by equation (4.61). This condition implies also, with $e^A_i \mathrm{n}_A = 0$, that $e^0_i = 0$. We therefore define the completely spatial vierbein $\mathfrak{e}^a_i$, and the conjugate momenta to the spatial vierbein $\mathfrak{e}^a_i$ reads, with $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{1/2}$,

$$p^i_a \equiv \frac{\partial(\sqrt{h}\mathcal{L})}{\partial \dot{\mathfrak{e}}^a_i} = \frac{2\epsilon^3}{16\pi G_N}\sqrt{h}\left(\Omega^{ki}{}_\perp \mathfrak{e}_{ak} - \Omega_k{}^k{}_\perp \mathfrak{e}^i_a\right) \tag{4.96}$$

Let us now follow the analysis of [15], which rewrite each term of the action (4.78) using the torsion constraints, and see here the effects of fermions in each term. Let us first express here the spin connection components that we will use in the following in terms of the vierbein using the torsion constraints (4.86)-(4.89) in the time gauge:

$$\Omega_{ikj} = -\mathfrak{e}_{kc}\partial_i \mathfrak{e}^c_j + \Gamma_{kij} - \left(\frac{i\kappa^2}{4\epsilon^3}\right)\bar{\Psi}\gamma^a{}_{bc}\mathfrak{e}^b_i \mathfrak{e}^c_k \mathfrak{e}_{ja}\Psi, \tag{4.97}$$

$$\begin{aligned}\Omega_{tij} = {} & \dot{\mathfrak{e}}^a_{[i}\mathfrak{e}_{j]a} + N^k \mathfrak{e}^a_{[i}\partial_{j]}\mathfrak{e}_{ka} + \mathfrak{e}^a_{[i}\partial_{j]}N^k \mathfrak{e}_{ka} + N^k \partial_{[j}\mathfrak{e}^a_{i]}\mathfrak{e}_{ka} - \epsilon N\left(\frac{i\kappa^2}{4\epsilon^3}\right)\bar{\Psi}\gamma_{0bc}\mathfrak{e}^b_i \mathfrak{e}^c_j \Psi \\ & - N^k\left(\frac{i\kappa^2}{4\epsilon^3}\right)\bar{\Psi}\gamma^a{}_{bc}\mathfrak{e}^b_i \mathfrak{e}^c_j \mathfrak{e}_{ak}\Psi,\end{aligned} \tag{4.98}$$

$$\Omega_{[ij]\perp} = -\left(\frac{i\kappa^2}{4\epsilon^3}\right)\bar{\Psi}\gamma_{0bc}\mathfrak{e}^b_i \mathfrak{e}^c_j \Psi, \tag{4.99}$$

$$\Omega_{[ij]k} = \partial_{[i}\mathfrak{e}^a_{j]}\mathfrak{e}_{ka} - \left(\frac{i\kappa^2}{4\epsilon^3}\right)\bar{\Psi}\gamma_{bc}{}^a \mathfrak{e}^b_i \mathfrak{e}^c_j \mathfrak{e}_{ak}\Psi, \tag{4.100}$$

where we have also rewritten the spatial vierbein postulate. The rewriting of the first term of the action (4.78) is given, upon integration by parts, as

$$\epsilon^3 \int \frac{dtd^3x}{8\pi G_N} \sqrt{h}\; \mathrm{n}_{[B} e^i_{A]} \mathcal{R}_{ti}{}^{AB} = \int dtd^3x\; p^i_a \dot{\mathfrak{e}}^a_i - \epsilon^3 \int \frac{dtd^3x}{16\pi G_N} \sqrt{h} \left(\Omega_t{}^{ij} \mathcal{T}_{ij\perp} - 2\Omega_t{}^i{}_\perp \mathcal{T}_{ij}{}^j\right). \quad (4.101)$$

Without the coupling to matter, the torsion constrains equal zero and the last integral disappears. This is no longer the case in the theory that we consider here. Imposing the torsion constraints (4.97)-(4.100), this term can be rewritten as

$$\begin{aligned} \epsilon^3 \int \frac{dtd^3x}{8\pi G_N} \sqrt{h}\; \mathrm{n}_{[B} e^i_{A]} \mathcal{R}_{ti}{}^{AB} &\approx \int dtd^3x\; \pi^{ij} \dot{h}_{ij} \\ &+ \epsilon^3 \int \frac{dtd^3x\sqrt{h}}{16\pi G_N} \left(\frac{1}{2} \epsilon N \left(\frac{i\kappa^2}{2\epsilon^3}\right)^2 \bar{\Psi}\gamma_0{}^{bc}\Psi\bar{\Psi}\gamma_{0bc}\Psi \right. \\ &\left. + \frac{1}{2} N^k \left(\frac{i\kappa^2}{2\epsilon^3}\right)^2 \bar{\Psi}\gamma^{abc}\mathfrak{e}_{ka}\Psi\bar{\Psi}\gamma_{0bc}\Psi - \partial_j N_i \left(\frac{i\kappa^2}{2\epsilon^3}\right) \bar{\Psi}\gamma_{0bc}\mathfrak{e}^{ib}\mathfrak{e}^{jc}\Psi \right), \end{aligned} \quad (4.102)$$

where the symbol $\approx$ is to stress that we imposed the torsion constraints. We also defined

$$\pi^{ij} \equiv \frac{\epsilon^3\sqrt{h}}{16\pi G_N} \left(\Omega^{(ij)}{}_\perp - \Omega_k{}^k{}_\perp h^{ij}\right). \quad (4.103)$$

The second term in the action, $\epsilon N e^i_{[A} e^j_{B]} \mathcal{R}_{ij}{}^{AB}$, splits into two parts using the time gauge condition $\mathrm{n}^A = \delta^A_0$, and the definition of the Carrollian metric (4.60)

$$\begin{aligned} &\epsilon^4 \int \frac{dtd^3x\sqrt{h}}{8\pi G_N} \left[N e^i_{[A} e^j_{B]} \left(\partial_i \Omega_j{}^{AB} + \Omega_i{}^{AC} \Omega_j{}^{DB} \eta_{CD}\right)\right] \\ &= \epsilon^4 \int \frac{dtd^3x\sqrt{h}}{8\pi G_N} N \left[e^i_{[A} e^j_{B]} \left(\partial_i \Omega_j{}^{AB} + \Omega_i{}^{AC} \Omega_j{}^{DB} \zeta_{CD}\right) + e^i_{[A} e^j_{B]} \Omega_i{}^A{}_\perp \Omega_j{}^B{}_\perp \right]. \end{aligned} \quad (4.104)$$

The first term constitutes the intrinsic spatial curvature in pure gravity [15]. To obtain the modifications brought by the coupling to fermions, we take the problem in reverse and start from the expression of the spatial curvature in terms of the spatial Christoffel symbols $\Gamma^k{}_{ij}$

$${}^3R = h^{ij} R^k{}_{ikj} = h^{ij} \left(\partial_k \Gamma^k{}_{ij} - \partial_i \Gamma^k{}_{kj} + \Gamma^k{}_{kl}\Gamma^l{}_{ij} - \Gamma^k{}_{il}\Gamma^l{}_{kj}\right), \quad (4.105)$$

that we have already introduced in section 3.2, and we then applied the spatial vierbein postulate (4.94). This will give us the first term of equation (4.104) accompanied by additional fermionic terms. At the end of the day, this term reduces to

$$\begin{aligned} &\epsilon^4 \int \frac{dtd^3x\sqrt{h}}{8\pi G_N} N e^i_{[A} e^j_{B]} \left(\partial_i \Omega_j{}^{AB} + \Omega_i{}^{AC} \Omega_j{}^{DB} \zeta_{CD}\right) \\ &\qquad = \epsilon^4 \int \frac{dtd^3x\sqrt{h}}{16\pi G_N} N \left[{}^3R - \left(\frac{i\kappa^2}{4\epsilon^3}\right)^2 \bar{\Psi}\gamma^a{}_{bc}\Psi\bar{\Psi}\gamma_a{}^{bc}\Psi \right]. \end{aligned} \quad (4.106)$$

Using the definition of the Hamiltonian constraint of magnetic Carrollian gravity $\mathcal{H}_M = -\frac{\sqrt{h}}{16\pi G_M} {}^3R$, we retrieved the same term of the Hamiltonian that we have defined in (4.1), accompanied by terms quartic in fermions. The second term requires less effort:

$$
\begin{aligned}
\epsilon^4 \int \frac{dtd^3x\sqrt{h}}{16\pi G_N} N 2 e^i_{[A} e^j_{B]} \Omega_i{}^A{}_\perp \Omega_j{}^B{}_\perp &= \epsilon^4 \int \frac{dtd^3x\sqrt{h}}{16\pi G_N} N 2 \Omega_i{}^{[i}{}_\perp \Omega_j{}^{j]}{}_\perp \\
&= \epsilon^4 \int \frac{dtd^3x\sqrt{h}}{16\pi G_N} N \left(\Omega_i{}^i{}_\perp \Omega_j{}^j{}_\perp - \Omega^{(ij)}{}_\perp \Omega_{(ji)\perp} - \Omega^{[ij]}{}_\perp \Omega_{[ji]\perp}\right) \\
&= -\frac{16\pi G_N}{\epsilon^2} \int dtd^3x \frac{N}{\sqrt{h}} \left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) \\
&\quad + \frac{\epsilon^4}{16\pi G_N} \int dtd^3x \sqrt{h} N \left(\frac{i\kappa^2}{4\epsilon^3}\right)^2 \bar{\Psi}\gamma_{0bc}\Psi\bar{\Psi}\gamma_0{}^{bc}\Psi,
\end{aligned}
\tag{4.107}
$$

where we have used the expression of $\Omega^{[ij]}{}_\perp$ obtained via the torsion constraints (c.f. (4.100)) and the definition of the conjugate momentum (4.103). Finally, the last term in (4.78) reads

$$
\begin{aligned}
-\epsilon^3 \int \frac{dtd^3x\sqrt{h}}{8\pi G_N} & \left(N^i e^j_A \mathrm{n}_B - N^j e^i_A \mathrm{n}_B\right) \left(\partial_i \Omega_j{}^{AB} + \Omega_i{}^{AC}\Omega_j{}^{DB}\eta_{CD}\right) \\
&= \epsilon^3 \int \frac{dtd^3x\sqrt{h}}{16\pi G_N} \Big[2\nabla_i N^i \Omega_j{}^j{}_\perp - 2\nabla_i N_j \Omega^{(ji)}{}_\perp - 2\nabla_i N_j \Omega^{[ji]}{}_\perp \\
&\qquad + 2N^i \left(\frac{i\kappa^2}{4\epsilon^3}\right) \bar{\Psi}\gamma^a{}_{bc}\mathfrak{e}^b_i \mathfrak{e}^{jc}\mathfrak{e}^k_a \Psi\ \Omega_{[jk]\perp}\Big] \\
&= 2\int dtd^3x\ N_i \nabla_j \pi^{ij} + 2\epsilon^3 \int \frac{dtd^3x\sqrt{h}}{16\pi G_N} \Big[\left(\frac{i\kappa^2}{4\epsilon^3}\right) \nabla_i N_j \bar{\Psi}\gamma_{0bc}\mathfrak{e}^{jb}\mathfrak{e}^{ic}\Psi \\
&\qquad - N^i \left(\frac{i\kappa^2}{4\epsilon^3}\right)^2 \bar{\Psi}\gamma^a{}_{bc}\mathfrak{e}^b_i \Psi\bar{\Psi}\gamma_0{}^c{}_a\Psi\Big]
\end{aligned}
\tag{4.108}
$$

The first equality is obtained by an integration by parts, followed by the repetitive uses of the torsion constraints and in particular the one associated with the vierbein postulate, which was used to construct the spatial covariant derivatives $\nabla_i N^j = \partial_i N^j + \Gamma^j{}_{ik}N^k$. The property $\partial_i\sqrt{h} = \sqrt{h}\Gamma^j{}_{ji}$ was also used. The second equality is obtained via an integration by parts of the two first terms of the first equality, the definition of the conjugate momentum (4.103), and the use of the torsion constraint associated with $\Omega^{[ij]}{}_\perp$. Remark that here the integration by parts does not produce a term proportional to the torsion, as the covariant derivative is defined in terms of the Levi-Civita connection. Let us now move on to the rewriting of the Dirac action (4.79). In the time gauge, it takes the form

$$
\begin{aligned}
S_{1/2} = \int dtd^3x\sqrt{h} \Big[&-\frac{i}{2}\bar{\Psi}\gamma_0\partial_t\Psi + \frac{i}{2}\bar{\Psi}\gamma^a\mathfrak{e}^i_a\epsilon N\partial_i\Psi + \frac{i}{2}\bar{\Psi}\gamma_0 N^i\partial_i\Psi \\
&+ \frac{i}{2}\bar{\Psi}\overleftarrow{\partial}_t\gamma_0\Psi - \frac{i}{2}\bar{\Psi}\overleftarrow{\partial}_i\gamma^a\mathfrak{e}^i_a\epsilon N\Psi - \frac{i}{2}\bar{\Psi}\overleftarrow{\partial}_i\gamma_0 N^i\Psi + \frac{i}{4}\bar{\Psi}\left(\Omega_t{}^{ij}\mathfrak{e}^c_i\mathfrak{e}^d_j\gamma_{0cd}\right. \\
&\left. - \epsilon N\mathfrak{e}^i_a\Omega_i{}^{jk}\mathfrak{e}^c_j\mathfrak{e}^d_k\gamma^a{}_{cd} - 2\epsilon N\mathfrak{e}^{ia}\Omega_{[ik]\perp}\mathfrak{e}^{kd}\gamma_{a0d} - N^i\Omega_i{}^{jk}\mathfrak{e}^c_j\mathfrak{e}^d_k\gamma_{0cd}\right)\Psi\Big].
\end{aligned}
\tag{4.109}
$$

The definition of the three-dimensional covariant spinor derivative is

$$
\nabla_i\Psi = \left(\partial_i - \frac{1}{4}\Omega_i{}^{CD}\gamma_{CD}\right)\Psi, \quad \bar{\Psi}\overleftarrow{\nabla}_i = \bar{\Psi}\left(\overleftarrow{\partial}_i + \frac{1}{4}\Omega_i{}^{CD}\gamma_{CD}\right), \tag{4.110}
$$

and in the time gauge

$$\nabla_i \Psi = \left(\partial_i - \frac{1}{4}\left(\Omega_i{}^{jk}\mathfrak{e}^c_j\mathfrak{e}^d_k\gamma_{cd} - 2\Omega_{[ik]\perp}\mathfrak{e}^{kd}\gamma_{0d}\right)\right), \tag{4.111}$$

$$\bar{\Psi}\overleftarrow{\nabla}_i = \bar{\Psi}\left(\overleftarrow{\partial}_i + \frac{1}{4}\left(\Omega_i{}^{jk}\mathfrak{e}^c_j\mathfrak{e}^d_k\gamma_{cd} + 2\Omega_{[ik]\perp}\mathfrak{e}^{kd}\gamma_{0d}\right)\right). \tag{4.112}$$

Using the torsion constraints to express the spin connections in terms of the vierbein and this definition in (4.109), we obtain

$$\begin{aligned} S_{1/2} = \int dtd^3x\sqrt{h}\Big[& \pi_\Psi\dot{\Psi} + \dot{\bar{\Psi}}\bar{\pi}_\Psi + \frac{i}{2}\bar{\Psi}\gamma_0 N^i\nabla_i\Psi - \frac{i}{2}\bar{\Psi}\overleftarrow{\nabla}_i N^i\gamma_0\Psi + \frac{i}{2}\bar{\Psi}\gamma^a\mathfrak{e}^i_a\epsilon N\nabla_i\Psi \\ & - \frac{i}{2}\bar{\Psi}\epsilon N\overleftarrow{\nabla}_i\gamma^a\mathfrak{e}^i_a\Psi + \frac{i}{4}\bar{\Psi}\mathfrak{e}^{ic}\mathfrak{e}^{jd}\gamma_{0cd}\Psi\left(\dot{\mathfrak{e}}_{[i}{}^a\mathfrak{e}_{j]a} + N^k\mathfrak{e}_{[i}{}^a\partial_{j]}\mathfrak{e}_{ka} + \mathfrak{e}_{[i}{}^a\partial_{j]}N^k\mathfrak{e}_{ka} + N^k\partial_{[j}\mathfrak{e}_{i]}{}^a\mathfrak{e}_{ka}\right) \\ & + \frac{i}{4}\left(-3\epsilon N\left(\frac{i\kappa^2}{4\epsilon^3}\right)\bar{\Psi}\gamma_0{}^{bc}\Psi\bar{\Psi}\gamma_{0bc}\Psi + \epsilon N\left(\frac{i\kappa^2}{4\epsilon^3}\right)\bar{\Psi}\gamma^a{}_{cd}\Psi\bar{\Psi}\gamma_a{}^{cd}\Psi\right)\Big], \end{aligned} \tag{4.113}$$

with the conjugate momentum to Ψ and its hermitian conjugate

$$\pi_\Psi \equiv -\frac{i}{2}\sqrt{h}\bar{\Psi}\gamma_0, \quad \bar{\pi}_\Psi \equiv \frac{i}{2}\sqrt{h}\gamma_0\Psi, \tag{4.114}$$

where we have introduced the notation $\dot{\bar{\Psi}} = \bar{\Psi}\overleftarrow{\partial}_t$. Remark here that there is a contribution in terms of the derivative of the vierbeins $\sqrt{h}\frac{i}{4}\left(\bar{\Psi}\mathfrak{e}^{ic}\gamma_{0ca}\Psi\right)\dot{\mathfrak{e}}^a_i$. The same term has been found in [33], which treats the Hamiltonian formulation of general relativity coupled to spin-1/2 fermions. In the latter paper, this term is justified as an extra contribution to the gravitational momenta due to the coupling, such that it plays the same role as the potential A_μ in the momentum of a particle subject to a magnetic field $p_\mu - eA_\mu = m\dot{x}_\mu$.

We will now regroup the two actions together. Using $\kappa^2 = 8\pi G_N$, and rescaling Newton's constant as $G_N = \epsilon^4 G_M$, we arrive at

$$S = \int dtd^3x\left[\dot{h}_{ij}\pi^{ij} + p^i_{Da}\dot{\mathfrak{e}}^a_i + \pi_\Psi\dot{\Psi} + \dot{\bar{\Psi}}\bar{\pi}_\Psi - N\mathcal{H}_\perp - N_i\mathcal{H}^i\right], \tag{4.115}$$

with

$$\pi^{ij} = \frac{\sqrt{h}}{16\pi\epsilon G_M}\left(\Omega^{(ij)}{}_\perp - \Omega_k{}^k{}_\perp h^{ij}\right), \quad p^i_{Da} = \frac{i}{4}\sqrt{h}\left(\bar{\Psi}\mathfrak{e}^{ic}\gamma_{0ca}\Psi\right) \tag{4.116}$$

$$\pi_\Psi = -\frac{i}{2}\sqrt{h}\bar{\Psi}\gamma_0, \quad \bar{\pi}_\Psi = \frac{i}{2}\sqrt{h}\gamma_0\Psi, \tag{4.117}$$

and

$$\begin{aligned} \mathcal{H}_\perp &= \mathcal{H}_M + \frac{16\epsilon^2\pi G_M}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - \epsilon\sqrt{h}\left(\frac{i}{2}\bar{\Psi}\gamma^a\mathfrak{e}^i_a\nabla_i\Psi - \frac{i}{2}\bar{\Psi}\overleftarrow{\nabla}_i\gamma^a\mathfrak{e}^i_a\Psi\right) \\ &\quad + \frac{\epsilon^2\sqrt{h}\pi G_M}{4}\left[\bar{\Psi}\gamma^a{}_{bc}\Psi\bar{\Psi}\gamma_a{}^{bc}\Psi - 3\bar{\Psi}\gamma_0{}^{bc}\Psi\bar{\Psi}\gamma_{0bc}\Psi\right] \\ \mathcal{H}^i &= -2\nabla_j\pi^{ij} - \sqrt{h}\,h^{ij}\left(\frac{i}{2}\bar{\Psi}\gamma_0\nabla_j\Psi - \frac{i}{2}\bar{\Psi}\overleftarrow{\nabla}_j\gamma_0\Psi\right) + \frac{i}{4}\partial_j\left(\sqrt{h}\bar{\Psi}\mathfrak{e}^{ic}\mathfrak{e}^{jb}\gamma_{0bc}\Psi\right). \end{aligned} \tag{4.118}$$

We have denoted here p^i_{Da} for the extra-contribution to the gravitational momenta due to the coupling with the Dirac fermions. The total Hamiltonian $\mathcal{H} = N\mathcal{H} + N^i\mathcal{H}_i$ contains the same terms as the one found in [33] (without the mass term), plus terms quartic in fermions. This difference, as already anticipated in the latter paper, is quite consistent with the fact that here we started from a first order formalism of general relativity, and then used torsion constraints to rewrite the spin connections in terms of vierbeins (except for one), whereas in the paper in question they started directly from a second-order formalism. We have indeed seen in section 4.1.3 that the switch between first and second order formalism leads to terms in the action quartic in the spinor fields.

Let us now return to the scaling in ϵ in the rescaled vierbein and spin connection (4.68). To take the Carrollian limit, one has to reinstate the parameter ϵ hidden in the time components of the spin connection and the vierbein. Concerning the vierbein, thanks to the time gauge, we have now $e^0_i = 0$ and $\mathrm{n}^A = n^A = \delta^A_0$. The object n^A is thus already equal to a Carrollian object, while the rescaled part of $e^A_i \equiv E^A_i$ is equal to zero. On the other hand, the spin connection that has not been expressed in terms of vierbein, $\Omega^{(ij)}{}_\perp$, must be expressed in terms of quantities that do not scale in the Carrollian limit:

$$\pi^{ij} = -\frac{\sqrt{h}}{16\pi G_M}\left(\omega^{(ij)} - h^{ij}\omega_k{}^k\right). \tag{4.119}$$

Therefore, we see that the conjugate momentum π^{ij} does not depend on ϵ when written in terms of the Carrollian spin connection. Taking this into account in the $\epsilon \to 0$ limit, we obtain the following Carrollian limit of the Einstein action coupled with massless Dirac fermions

$$S_{Carroll} = \int dt d^3x \left[\dot{h}_{ij}\pi^{ij} + p^i_{Da}\dot{\mathfrak{e}}^a_i + \pi_\Psi\dot{\Psi} + \dot{\bar{\Psi}}\bar{\pi}_\Psi - N\mathcal{H}_\perp - N_i\mathcal{H}^i\right], \tag{4.120}$$

with

$$\pi^{ij} = -\frac{\sqrt{h}}{16\pi G_M}\left(\omega^{(ij)} - h^{ij}\omega_k{}^k\right), \quad p^i_{Da} = \frac{i}{4}\sqrt{h}\left(\bar{\Psi}\mathfrak{e}^{ic}\gamma_{0ca}\Psi\right), \tag{4.121}$$

$$\pi_\Psi = -\frac{i}{2}\sqrt{h}\bar{\Psi}\gamma_0, \quad \bar{\pi}_\Psi = \frac{i}{2}\sqrt{h}\gamma_0\Psi, \tag{4.122}$$

and

$$\begin{aligned}
&\mathcal{H}_\perp = \mathcal{H}_M\\
&\mathcal{H}^i = -2\nabla_j\pi^{ij} - \sqrt{h}\; h^{ij}\left(\frac{i}{2}\bar{\Psi}\gamma_0\nabla_j\Psi - \frac{i}{2}\bar{\Psi}\overleftarrow{\nabla}_j\gamma_0\Psi\right) + \frac{i}{4}\partial_j\left(\sqrt{h}\bar{\Psi}\mathfrak{e}^{ic}\mathfrak{e}^{jb}\gamma_{0bc}\Psi\right).
\end{aligned} \tag{4.123}$$

Let us try to analyze this limit. Firstly, one can notice that all the quartic terms in the Dirac fields vanish. We can also see that the Dirac fields have no impact in the limit on the $\mathcal{H}_\perp$ part of the Hamiltonian, as we retrieved the same magnetic limit as for pure gravity for this specific term. An interesting observation is that all the terms containing spatial covariant derivatives of the form $\mathfrak{e}^i_a\nabla_i\Psi$ vanish too. This feature is also found in the Carrollian limit of the Dirac action discussed in [16], which in addition to presenting the Carrollian action of gravity found from a gauging of the Carroll algebra, presents the Carrollian action of the matter action of the Dirac field in the first order formalism, the scalar field and electromagnetism via the same method. With the method used to obtain the limit (4.120), which considers the same rescaling of the vierbein and the spin connection as in [16], these two limits should be the same. We hope to carry out the explicit check in the future[f].

With the Carrollian action (4.115), we have therefore reached our main goal, which was to describe gravity coupled to fermionic matter in the Carrollian limit. This result also indicates that the scaling (4.68) is well adapted to describe pure gravity or coupled to fermionic matter. If one refers to the terminology used in [11], this limit is the *magnetic* limit of the Einstein-Cartan action coupled to massless Dirac fermions.

4.4.3 Carrollian limit of gravity with two-component spinors

Now we can go further and use our expertise of the Carrollian limit of spin-1/2 massless fermions in the free case. Indeed, let us recall that in the analysis of the Carrollian limit of the Dirac action performed in section 2.2.3, we have noticed that the rescaling of the fermionic fields, such as the kinetic term of the action is not impacted in the limit, plays an important role. This analysis was fruitful only if we decomposed the whole field into its two-components spinors ϕ and χ. We will do the same here and observe the impact on the action in the limit.

We start from the action (4.115), decompose the field Ψ into two-component spinor $\Psi = \begin{pmatrix}\phi \\ \chi\end{pmatrix}$, and use the Pauli-Dirac representation of the gamma matrices. Then, we apply the rescaling

$$\chi \to \epsilon\chi, \quad \pi_\chi \to \frac{1}{\epsilon}\pi_\chi. \tag{4.124}$$

This rescaling is the same as the one used in the free case, and it does not impact the kinetic terms of the action. In the following, as it is customary when the Dirac action is written in an antisymmetric form, as we have considered in our analysis, we will not replace the fields by the conjugate moments in the Hamiltonian part of the action (see, e.g., [33] or [40]), and so the rescaling takes the form

$$\chi \to \epsilon\chi, \quad \chi^\dagger \to \frac{1}{\epsilon}\chi^\dagger. \tag{4.125}$$

Taking the $\epsilon \to 0$ limit, the Carrollian action with rescaling of the two-component spinor fields is given by

$$S_{Carroll} = \int dtd^3x \left[\dot{h}_{ij}\pi^{ij} + p^i_{Da}\dot{\mathfrak{e}}^a_i + \pi_\phi\dot{\phi} + \pi_\chi\dot{\chi} + \dot{\phi}^\dagger\pi^\dagger_\phi + \dot{\chi}^\dagger\pi^\dagger_\chi - N\mathcal{H}_\perp - N_i\mathcal{H}^i\right], \tag{4.126}$$

with

$$\pi^{ij} = -\frac{\sqrt{h}}{16\pi G_M}\left(\omega^{(ij)} - h^{ij}\omega_k{}^k\right), \quad p^i_{Da} = \frac{i}{4}\sqrt{h}\left[\phi^\dagger\sigma_{ca}\phi + \chi^\dagger\sigma_{ca}\chi\right]\mathfrak{e}^{ic}, \tag{4.127}$$

$$\pi_\phi = \frac{i\sqrt{h}}{2}\phi^\dagger, \quad \pi_\chi = \frac{i\sqrt{h}}{2}\chi^\dagger, \tag{4.128}$$

[f] Note that in [16], the Dirac action defined with the Lagrangian (4.33) is used. This action is inequivalent to the one we used, and this should be taken into account in the comparison.

and their hermitian conjugates

$$\pi_\phi^\dagger = -\frac{i\sqrt{h}}{2}\phi, \quad \pi_\chi^\dagger = -\frac{i\sqrt{h}}{2}\chi. \tag{4.129}$$

The constraints take the form

$$\begin{aligned}
\mathcal{H}_\perp &= \mathcal{H}_M - \sqrt{h}\left(\frac{i}{2}\chi^\dagger\sigma^a\mathfrak{e}_a^i\nabla_i\phi - \frac{i}{2}\chi^\dagger\overleftarrow{\nabla}_i\mathfrak{e}_a^i\sigma^a\phi\right) + \frac{\sqrt{h}\pi G_M}{4}\left(\chi^\dagger\sigma^{abc}\phi\chi^\dagger\sigma_{abc}\phi\right), \\
\mathcal{H}^i &= -2\nabla_j\pi^{ij} + \sqrt{h}\; h^{ij}\left(\frac{i}{2}\left(\phi^\dagger\nabla_j\phi + \chi^\dagger\nabla_j\chi\right) - \frac{i}{2}\left(\phi^\dagger\overleftarrow{\nabla}_j\phi + \chi^\dagger\overleftarrow{\nabla}_j\chi\right)\right) \\
&+ \frac{i}{4}\partial_j\left(\sqrt{h}\left(\phi^\dagger\sigma^{bc}\phi + \chi^\dagger\sigma^{bc}\chi\right)\mathfrak{e}_c^i\mathfrak{e}_b^j\right).
\end{aligned} \tag{4.130}$$

We have defined $\sigma_{abc} \equiv \frac{1}{4}\{\sigma_a, [\sigma_b, \sigma_c]\}$. Let us try to analyze this limit and to compare it with the one obtained without rescaling of the spinor fields at expression (4.120). The major difference takes place within $\mathcal{H}_\perp$, where now $\mathcal{H}_M$ is not the only term that remains in the limit. Among the persistent terms, there are spatial covariant derivatives of fields of the form $\mathfrak{e}_a^i\nabla_i\phi$. We can therefore see that rescaling the two-component spinor fields ϕ and χ, and only then taking the limit allows us to keep the terms in the spatial derivatives of the Dirac fields which disappeared in the limit without rescaling. This is the same behavior that we observed in the Carrollian limit of the free massless Dirac action that we have analyzed in section 2.2.3, where one of the two spatial gradients present in the action persisted in the limit with rescaling, while both vanished without rescaling. One can also notice that, while they all vanish without rescaling, one quartic fermionic term is still present in the limit with rescaling of the fields. By considering the limit with rescaling of the two-components spinors of the Dirac field, not only do we retain all the terms present in the limit without rescaling, but we also keep additional terms. Among the additional terms, there are in particular spatial covariant derivatives of the fermionic fields.

We can therefore conclude with the fact that rescaling the two-components spinor fields allows us to obtain a richer Carrollian theory, containing in particular terms in the spatial derivatives of the fields, which is not the case in the limit without rescaling. Since we obtained the same conclusion in the flat case, we can conclude that the latter property is applicable in flat or curved space.

4.4.4 Summary of the results

Since this section has mixed many concepts, and since many different calculations and results have been presented, we would like to summarize the original results derived in this section:

1. As was done for the Einstein-Cartan action in [15], the massless Dirac action coupled to a curved background is rewritten in the $3+1$ decomposition of space-time using the relation between the first order and the Hamiltonian formalism of general relativity. This was already studied by [33], while the rescaling of the vierbein and the spin connection (4.68) was not considered before in this context.

2. Starting from the Einstein-Cartan action coupled to the massless Dirac action in $3+1$ form, the torsion constraints obtained by the variation with respect to the normal and tangential components to the hypersurface of the spin connections are derived.

3. Using the time-gauge, the latter action is then rewritten in Hamiltonian form using the torsion constraints. The final form is given by the expression (4.115).

4. The direct $\epsilon \to 0$ limit of this action is then computed. This gives the action describing the coupling of Carrollian gravity to massless Dirac fermions (4.120). In the limit, terms of the form $\mathfrak{e}^i_a \nabla_i \Psi$ vanish, as well as all the terms quartic in the Dirac fields.

5. Inspired by the decomposition in terms of the two-components spinor fields ϕ and χ used in the free case (see section 2.2.3), the Hamiltonian action (4.115) is expressed in terms of these two fields using the decomposition of the Dirac spinor $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$. The rescaling of the fields (4.124) is used, and only then the $\epsilon \to 0$ limit is taken. By considering the limit with rescaling of the two-components spinors of the Dirac field, not only do all the terms present in the limit without rescaling are present, but additional terms are kept. In particular, this allows keeping terms of the form $\mathfrak{e}^i_a \nabla_i \phi$, whereas, in the limit without rescaling, all the spatial derivatives vanish. The same behavior is found in the Carrollian limit of the free Dirac action using the same rescaling of the fields.

Conclusion and prospects

Fermions are necessary to describe fundamental interactions and therefore, in view of the recent applications of Carrollian physics, it is important to understand how they behave in the limit of vanishing speed of light. This was the main motivation of this thesis. In the following, the main results that we achieved in this work are summarized, before discussing some possible further developments.

After some reminders and definition of concepts that are necessary for the comprehension of this thesis in chapter 1, we began the study of possible inequivalent Carrollian limits of the Dirac theory in chapter 2. As a first step, the massless Dirac fields were considered. The goal was to extend the analysis of [11], which has defined the "electric" and "magnetic" Carrollian limits for Lorentz-invariant bosonic theories through their Hamiltonian formulation, to the case of Dirac fermions. To reach this aim, it has been understood that the decomposition of the Dirac field into its two component spinors ϕ and χ was playing a key role. This idea of using two-components spinors to study the Carrollian limit of a fermionic theory has already been used in [14], where the "ultra-relativistic Dirac equations" were obtained. Without this decomposition, the direct Carrollian limit (the magnetic one, if one refers to the terminology of [11]) in our analysis leads to a vanishing Hamiltonian, and no rescaling of the canonical variables with the speed of light c to avoid that was possible. However, performing the analysis with this decomposition into two-components spinors allowed performing a rescaling of the fields so as to obtain a non-trivial Carrollian limit. The equations of motions of the resulting Carrollian action coincide with the "ultra-relativistic Dirac equations" obtained in [14]. This was the main result of this chapter.

In chapter 3, a brief introduction of the Hamiltonian (ADM) formulation of general relativity was presented. In this introduction, it is intended to give the key steps to understanding the reformulation of the gravitational action into a Hamiltonian form, which is the starting point of chapter 4. In this chapter, the aim was to couple Dirac fermions to Carrollian gravity, thus extending to matter coupling the results of [15]. As a recall, this paper aimed to clarify the links between the Hamiltonian analysis of [11] to obtain the magnetic limit of Einstein's theory and the definition of the Carrollian gravity given in [16] through a gauging of the Carroll algebra. To reach our goal, the relation between the Hamiltonian and first-order formulations of general relativity is used, along the lines of the analysis of [15]. As a first step, the massless Dirac action coupled to a curved background was rewritten in "$3+1$" form, using the same rescaling of the vierbein in ϵ (where $\epsilon \to 0$ define the Carrollian limit in this case) than in [16] and [15]. Then, following the analysis of [15], the rewriting of the Einstein-Cartan action coupled to massless Dirac fermions in Hamiltonian form was performed. Taking the direct $\epsilon \to 0$ limit, we obtained the *magnetic Carrollian limit of Einstein's theory coupled to massless Dirac* *fermions*. The analysis was then taken a step further, using our expertise in the Carrollian limit of free Dirac theory. Having in mind that in that case, the decomposition of the field into two component spinor fields plays an important role, it is applied to the coupling case. With the same rescaling of the fields than in the free case, and then taking the $\epsilon \to 0$ limit, not only all the terms present in the magnetic limit without rescaling are present, but additional terms are also kept in the limit. We have then concluded with the fact that the decomposition of the field into two-components spinor fields, combined with the rescaling of these fields, allows us to obtain a richer Carrollian theory, whether we consider the free case or the coupling to gravity.

Finally, let us discuss some perspectives for future research. A natural extension of this work would be to study generic fermionic fields by extending the analysis performed in this thesis to the spin-3/2 Rarita-Schwinger field, which is important because these are the basic building blocks of supergravity theories, and its higher-spin counterparts. Another possible outlook is that, as in the free case where we have computed the Poisson bracket $\{\mathcal{E}(x), \mathcal{E}(x')\}$ to check that the theory is Carroll invariant, we would like to explicitly compute the Poisson bracket of the Hamiltonian constraints $\mathcal{H}_\perp$ and $\mathcal{H}_i$ of the Carrollian theories obtained in the coupling part of this thesis. Another direction could be to look for the electric Carrollian limit of Einstein's theory coupled to massless Dirac fermions. As discussed in [15], it is not so obvious to recover the electric theory via a rescaling of the vierbein and the spin connection. A possible perspective would therefore be to find of a way to realize this limit, and then include the coupling to fermions. Finally, one could extend the analysis performed in this thesis to the massive fermionic case. This choice to focus on the massless case in this thesis is motivated by the fact that the free massless Dirac theory is invariant under conformal transformations. This setup is expected to find applications in flat-space holography, where the boundary theory is expected to be invariant under the conformal extension of the Carroll group.

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Appendix A

Proof of the Carroll invariance conditions

This appendix is dedicated to the proof of the Carroll invariance conditions for the particular case of a scalar field. We follow the demonstration given in [11]. To begin, we first recall some notions about Noether currents and charges, based on [28] and [37].

Let $\varphi^i(x)$ denote a set of scalar fields with Lagrangian density $\mathcal{L}(x) = \mathcal{L}(\varphi^i(x), \partial_\mu \varphi^i(x))$. Consider the infinitesimal transformation of the fields $\varphi^i(x) \to \varphi^i(x) + \delta\varphi^i(x)$, such as $\mathcal{L}(x) \to \mathcal{L}(x) + \delta\mathcal{L}(x)$, where $\delta\mathcal{L}(x)$ is given by

$$\delta\mathcal{L}(x) = \frac{\partial\mathcal{L}}{\partial\varphi^i(x)}\delta\varphi^i(x) + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi^i(x)\right)}\partial_\mu\delta\varphi^i(x). \tag{A.1}$$

Using the Euler-Lagrange equations, this can be rewritten as

$$\delta\mathcal{L}(x) = \partial_\mu\left(\frac{\partial\mathcal{L}(x)}{\partial\left(\partial_\mu\varphi^i(x)\right)}\delta\varphi^i(x)\right). \tag{A.2}$$

The object between parentheses is identified as the Noether current

$$j^\mu(x) \equiv \frac{\partial\mathcal{L}(x)}{\partial\left(\partial_\mu\varphi^i(x)\right)}\delta\varphi^i(x). \tag{A.3}$$

If a transformation of the fields such as $\delta\mathcal{L} = 0$, i.e., that leaves the Lagrangian invariant, can be found, one obtains the conservation of the Noether current $\partial_\mu j^\mu = 0$. Let us now consider the more general case where the variation of the Lagrangian is not zero, but is given by a total derivative $\partial_\mu K^\mu(x)$ for some $K^\mu(x)$. In that case, there is still a conserved current, but now given by

$$j^\mu(x) = \frac{\partial\mathcal{L}(x)}{\partial\left(\partial_\mu\varphi^i(x)\right)}\delta\varphi^i(x) - K^\mu(x). \tag{A.4}$$

This can be exemplified through the space-time translations of the fields $\varphi^i(x) \to \varphi^i(x+a)$, where a^μ is a constant four-vector. One has in that case $\mathcal{L}(x) \to \mathcal{L}(x+a)$. The infinitesimal version of this transformation is $\varphi^i(x) \to \varphi^i(x) + a^\nu\partial_\nu\varphi^i(x)$, and the variation of the Lagrangian under this transformation is given by $\delta\mathcal{L}(x) = -a^\nu\partial_\nu\mathcal{L}(x) = -\partial_\nu(a^\nu\mathcal{L}(x))$. One can therefore identify $K^\nu = -a^\nu \mathcal{L}$. The conserved current is therefore given by

$$\begin{aligned} j^\mu(x) &= \frac{\partial \mathcal{L}(x)}{\partial\left(\partial_\mu \varphi^i(x)\right)} a^\nu \partial_\nu \varphi^i(x) - a^\mu \mathcal{L}(x), \\ &= -a_\nu \mathcal{T}^{\mu\nu}(x), \end{aligned} \tag{A.5}$$

with $\mathcal{T}^{\mu\nu}$, the energy-momentum tensor, defined by

$$\mathcal{T}^{\mu\nu}(x) \equiv -\frac{\partial \mathcal{L}(x)}{\partial\left(\partial_\mu \varphi^i(x)\right)} \partial^\nu \varphi^i(x) + g^{\mu\nu} \mathcal{L}(x). \tag{A.6}$$

Carrying out the same demonstration for the case of Lorentz transformations $\varphi^i(x) \to \varphi^i(x + \omega \cdot x)$, the conserved current reads

$$\begin{aligned} j^\mu(x) &= \frac{1}{2}\omega_{\nu\rho}\left(x^\rho \mathcal{T}^{\mu\nu}(x) - x^\nu \mathcal{T}^{\mu\rho}(x)\right), \\ &= \frac{1}{2}\omega_{\nu\rho} \mathcal{M}^{\mu\nu\rho}, \end{aligned} \tag{A.7}$$

with $\omega_{\nu\rho} = -\omega_{\rho\nu}$, which parametrize the infinitesimal Lorentz transformations. The conserved charges $Q = \int d^3x\ j^0(x)$ associated with Lorentz transformations and space-time translations are

$$\begin{aligned} M^{\nu\rho} &= \int d^3x\ \mathcal{M}^{0\nu\rho}(x), \\ P^\mu &= \int d^3x\ \mathcal{T}^{0\mu}(x). \end{aligned} \tag{A.8}$$

The conserved charge P^μ associated with space-time translations can be split into its spatial and time components

$$P^k = \int d^3x\ \mathcal{T}^{0i} \equiv \int d^3x\ \mathcal{P}^k(x), \tag{A.9}$$

$$E = \int d^3x\ \mathcal{T}^{00} \equiv \int d^3x\ \mathcal{E}(x), \tag{A.10}$$

where E is the Hamiltonian, and P^k the momentum operator. If we restrict ourselves to the case of spatial translations and rotations, the vector $K^\mu(x)$ defined above has a vanishing time component. In this case, the Noether charge simplifies to

$$Q = \int d^3x\ j^0(x), \tag{A.11}$$

$$= \int d^3x\ \frac{\partial \mathcal{L}(x)}{\partial\left(\partial_0 \varphi^i(x)\right)} \delta\varphi^i(x), \tag{A.12}$$

$$= \int d^3x\ \pi_i(x) \delta\varphi^i(x), \tag{A.13}$$

where we have used the definition of the conjugate momentum

$$\pi_i(x) = \frac{\partial \mathcal{L}(x)}{\partial\left(\partial_0 \varphi^i(x)\right)}, \tag{A.14}$$

and the definition of the conserved current (A.4). Using the fact that

$$\delta\varphi^i(x) = \mathcal{L}_\xi \varphi^i(x), \tag{A.15}$$

under infinitesimal spatial translations and rotations parametrized by the vector ξ^μ, we have that

$$Q = \int d^3x \ \pi_i \mathcal{L}_\xi \varphi^i(x). \tag{A.16}$$

With this result, it is easy to see that the Poisson bracket of the Noether charge and the field is given by

$$\left\{\varphi^i(x), Q\right\} = \mathcal{L}_\xi \varphi^i(x). \tag{A.17}$$

One can also prove that we have equivalently

$$\left\{\pi^i, Q\right\} = \mathcal{L}_\xi \pi^i. \tag{A.18}$$

It is possible to prove that these results are also valid for time translations and Lorentz boosts. However, for the purpose of this work, we will only need spatial translations and translations, we thus restrict to this case where the manipulations are simpler. These results are the starting point of the demonstration of the Carroll invariance conditions presented in the next paragraphs following [11].

Let us first recall that, as mentioned in section 2.1, a necessary and sufficient condition for the theory to be Carroll invariant is that the generators obey the Carroll algebra given in (2.16). We have also noted that the requirement to fulfill the Carroll algebra implies conditions on the Poisson brackets of $\mathcal{P}_k(x)$ and $\mathcal{E}(x)$ since all the generators are constructed in terms of them. We concluded by saying that the non-trivial conditions for the theory to be Carroll invariant are, in fact, on $\mathcal{E}$. There are two of them: (1) $\mathcal{E}(x)$ have to be a scalar under spatial translations and rotations; and (2) $\{\mathcal{E}(x), \mathcal{E}(x')\} = 0$. We will prove this result in the following.

First, we focus on the conditions on the momentum density $\mathcal{P}_k$. As a recall, the expressions of the generators of spatial translations and rotations are,

$$P_k = \int d^dx \ \mathcal{P}_k(x), \quad M^{rs} = \int d^dx \ (x^r\delta^{sk} - x^s\delta^{rk})\mathcal{P}_k(x). \tag{A.19}$$

Specifying ξ^k to be a spatial translation ($\xi^k = a^k$) or a spatial rotation ($\xi^k(x) = \frac{1}{2}\omega_{rs}\left(x^r\delta^{sk} - x^s\delta^{rk}\right)$) in the Poisson brackets (A.17) and (A.18), one can see that the field and its conjugate momentum must satisfy the relations $\left\{\varphi^i(x), \int d^3y \ \xi^k(y)\mathcal{P}_k(y)\right\} = \mathcal{L}_\xi\varphi^i$ and $\left\{\pi^i, \int d^3y \ \xi^k(y)\mathcal{P}_k(y)\right\} = \mathcal{L}_\xi\pi^i$. Using these two latter Poisson brackets, or equivalently using the result (A.16), one can deduce that $\mathcal{P}_k(x)$ has to be given by $\int d^3x \ \xi^k(x)\mathcal{P}_k(x) = \int d^3x \ \pi_A\mathcal{L}_\xi\varphi^i(x)$. Similarly, these also imply in general $\left\{F(x), \int d^3y \ \xi^k(y)\mathcal{P}_k(y)\right\} = \mathcal{L}_\xi F(x)$, where $F(x)$ can be any function of the fields. One can therefore consider $\frac{1}{2}M^{rs}\omega_{rs} = \int d^3y \ \xi^k(y)\mathcal{P}_k(y)$ with $\xi^k(y) = \omega_{rs}y^r\delta^{sk}$ and

$$\begin{aligned}\left\{\mathcal{P}_k(x), \frac{1}{2}M^{rs}\omega_{rs}\right\} &= \left\{\mathcal{P}_k(x), \int d^3y \ \xi^k(y)\mathcal{P}_k(y)\right\} && \text{(A.20)}\\ &= \mathcal{L}_\xi\mathcal{P}_k(x). && \text{(A.21)}\end{aligned}$$

Integrating over space this Poisson bracket and performing an integration by part on the right-hand side of the last equality, one retrieves the exact expression for $\{P_k, M^{rs}\}$. The same steps can be applied for the Poisson bracket of the spatial rotations generators but with $\mathcal{M}^{0km} \equiv (x^k\delta^{ml} - x^m\delta^{kl})\mathcal{P}_l(x)$ instead of $\mathcal{P}_k(x)$ on the left-hand side of (A.21). The conditions on the Poisson bracket of $P_k(x)$ are therefore fulfilled without even knowing the action. We therefore see that the non-trivial conditions for Carroll invariance are indeed on $\mathcal{E}$. The first condition $\{\mathcal{E}(x), \mathcal{E}(x')\} = 0$ implies straightforwardly that $\{H, B_k\}$ and $\{B_k, B_m\}$ vanishes using the definition of H and B_k. For the second condition, if $\mathcal{E}(x)$ is a scalar under spatial translations and rotations, one has, using $\left\{F(x), \int d^3y\ \xi^k(y)\mathcal{P}_k(y)\right\} = \mathcal{L}_\xi F(x)$, that

$$\left\{\mathcal{E}(x), \int d^3y \left(a^k\mathcal{P}_k(y) + \omega^k_r y^r \mathcal{P}_k(y)\right)\right\} = \left(a^k + \omega^k_r x^r\right)\partial_k\mathcal{E}(x), \tag{A.22}$$

which can be equivalently rewritten

$$\left\{\mathcal{E}(x), a^k P_k(y) + \omega^k_r x^r P_k(y)\right\} = \left(a^k + \omega^k_r x^r\right)\partial_k\mathcal{E}(x). \tag{A.23}$$

Integrating over space and performing an integration by part on the right-hand side provides $\{H, P_k\}$ and $\{H, M^{rs}\}$ equal to zero, as request. For the last Poisson brackets to be found to have the full Carroll algebra, one starts by multiplying $\mathcal{E}(x)$ by x^m in (A.22). Then, integrating over space, and then performing an integration by parts on the right-hand side, provides finally the correct expression of $\{B^m, P_k\}$ and $\left\{B^k, M^{rs}\right\}$. This proves that the non-trivial conditions for the Carroll algebra to be fulfilled are: (1) $\mathcal{E}(x)$ has to be a scalar under spatial translations and rotations; (2) $\{\mathcal{E}(x), \mathcal{E}(x')\} = 0$.

As explained in the section 1.3 of [37] concerning the “Noether current and charges”, for various field systems, such as the Dirac field, the energy-momentum tensor $\mathcal{T}^{\mu\nu}$ derived from the Noether procedure is conserved but it is not symmetric. Nevertheless, it is possible to rectify this issue by modifying $\mathcal{T}^{\mu\nu}$ in all cases, thereby restoring its symmetry. This rectification is achieved in the case of the Dirac field by adding a total derivative, so the Noether charges are not modified. The derivation we have made in this appendix is therefore also applicable to Dirac fields according to these explanations. However, as an outlook, we would like to take a closer look at this derivation in the case of the Dirac field. For the purposes of this thesis, we assume that this derivation holds for Dirac fields.

Appendix B

Boundary terms in general relativity

This appendix is dedicated to the role of the boundary terms in general relativity.

We consider here an arbitrary region $\mathscr{V}$ of the space-time manifold, bounded by a closed hypersurface $\partial\mathscr{V}$. The precise action functional for general relativity is given by

$$S_G[g] = S_H[g] + S_B[g] - S_0, \tag{B.1}$$

where $S_H[g]$ is the Hilbert action, $S_B[g]$ is a boundary term, and S_0 a nondynamical term that affects the numerical value of the action but not the equations of motion. They are given by

$$S_H[g] = \frac{1}{16\pi G_N}\int_{\mathscr{V}} d^4x\sqrt{-g}\,\mathcal{R}, \tag{B.2}$$

$$S_B[g] = \frac{1}{8\pi G_N}\oint_{\partial\mathscr{V}} d^3x\sqrt{h}\,\varepsilon K, \tag{B.3}$$

$$S_0 = \frac{1}{8\pi G_N}\oint_{\partial\mathscr{V}} d^3x\sqrt{h}\,\varepsilon K_0, \tag{B.4}$$

with $\mathcal{R}$ the Ricci scalar in $\mathscr{V}$, K the trace of the extrinsic curvature of $\partial\mathscr{V}$, and h the determinant of the induced metric on $\partial\mathscr{V}$. Coordinates x^α are used in $\mathscr{V}$ and x^i in $\partial\mathscr{V}$. With the matter action taken to be of the form

$$S_M[\phi;g] = \int_{\mathscr{V}} d^4x\sqrt{-g}\,\mathcal{L}\left(\phi,\partial_\alpha\phi;g_{\alpha\beta}\right), \tag{B.5}$$

for some Lagrangian density $\mathcal{L}$, the complete action functional is given by

$$S[g;\phi] = \int_{\mathscr{V}} d^4x\sqrt{-g}\left(\frac{\mathcal{R}}{16\pi G_N}+\mathcal{L}\right) + \frac{1}{8\pi G_N}\oint_{\partial\mathscr{V}} d^3x\sqrt{h}\,\varepsilon\left(K-K_0\right). \tag{B.6}$$

The role of $S_B[g]$ lies in the variation of $S_G[g]$. If one performs the variation of the Hilbert term $S_H[g]$ with respect to $g^{\alpha\beta}$, with the fact that the variation is subject to the condition

$$\delta g_{\alpha\beta}\big|_{\partial\mathscr{V}} = 0, \tag{B.7}$$

one obtains the correct left-hand side of the Einstein field equations plus an additional boundary term. The reason for including $S_B[g]$ in the gravitational action is that its variation will cancel the boundary term obtained by varying the Hilbert term. The inclusion of the boundary term $S_B[g]$ in the gravitational action therefore permits obtaining, with the matter action, the Einstein field equation $\mathcal{G}_{\alpha\beta} = 8\pi G_N \mathcal{T}_{\alpha\beta}$, with $\mathcal{T}_{\alpha\beta}$ the stress energy tensor. The role of S_0 lies in the definition of the gravitational action for asymptotically flat space-times. Doing the calculation permits to observe that the gravitational action, and in particular the boundary term $S_B[g]$ for flat space-time, is infinite. This problem does not go away when the space-time is curved, and therefore this would imply that $S_G[g]$ is not well-defined for asymptotically flat space-times. This is remedied by the introduction of S_0 such that K_0 is chosen to be equal to the extrinsic curvature of $\partial\mathscr{V}$ embedded in flat space-time. This definition makes the difference $S_B[g] - S_0$ well-defined in flat space-time. More details are given in [17].

Appendix C

Integration by parts with torsion

This appendix is dedicated to the integration by parts in the presence of a connection containing torsion, and is taken from [37]. We prove here that there is an extra-term proportional to the torsion when one integrates by part in curved space-time with torsion.

We first recall that the components ${\Gamma^\rho}_{\mu\nu}$ of a connection compatible with the metric takes the form

$$
{\Gamma^\rho}_{\mu\nu} = {\Gamma^\rho}_{\mu\nu}[g] - {\mathcal{K}_{\mu\nu}}^{\rho}, \tag{C.1}
$$

with ${\Gamma^\rho}_{\mu\nu}[g] = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$, the Christoffel symbols of the Levi-Civita connection, and

$$
\mathcal{K}_{\mu\nu\rho} = -\frac{1}{2} \left(\mathcal{T}_{\mu\nu\rho} - \mathcal{T}_{\mu\rho\nu} - \mathcal{T}_{\nu\rho\mu} \right), \tag{C.2}
$$

the contorsion tensor, defined in section 4.1.3, and $\mathcal{T}_{\mu\nu\rho} = {\mathcal{T}_{\mu\nu}}^{\rho} g_{\sigma\rho}$. The first step in this demonstration is to prove that

$$
\nabla_\mu V^\mu = \partial_\mu V^\mu + {\Gamma^\nu}_{\nu\mu}[g] V^\mu + {\mathcal{T}_{\nu\mu}}^{\nu} V^\mu. \tag{C.3}
$$

Using the usual covariant derivative of a vector field and equation (C.1), we have

$$
\begin{aligned}
\nabla_\mu V^\mu &= \partial_\mu V^\mu + {\Gamma^\nu}_{\nu\mu}, &\text{(C.4)}\\
&= \partial_\mu V^\mu + {\Gamma^\nu}_{\nu\mu}[g] V^\mu - {\mathcal{K}_{\nu\mu}}^{\nu} V^\mu. &\text{(C.5)}
\end{aligned}
$$

Using the definition of the contorsion tensor (C.2), we arrive to

$$
\nabla_\mu V^\mu = \partial_\mu V^\mu + {\Gamma^\nu}_{\nu\mu} + \frac{1}{2} \left({\mathcal{T}_{\mu\nu}}^{\mu} - g^{\rho\mu} \mathcal{T}_{\mu\rho\nu} - {\mathcal{T}_{\nu\rho}}^{\rho} \right) V^\nu. \tag{C.6}
$$

The second term in the parentheses is equal to zero due to the anti-symmetry on the indices ρ, μ of the torsion tensor, and we use again the anti-symmetry of the torsion tensor in the last term. We finally arrive at the desired expression (C.3).

We will now use this expression to examine integration by parts in the presence of torsion. Usually, we have that in Minkowski space-time, total derivatives $\int d^3x \partial_\mu V^\mu = 0$ if $V^\mu(x)$ vanishes at large distance. This is the property that allows integration by parts. Let us now take a look at this property for the covariant derivative $\nabla_\mu V^\mu$. Using the property

$$\partial_\mu \sqrt{-g} = \sqrt{-g}\Gamma^\rho{}_{\rho\mu}(g), \tag{C.7}$$

and equation (C.3) that we have derived above, we have

$$\int d^3x \sqrt{-g}\ \nabla_\mu V^\mu = \int d^3x\ \left(\sqrt{-g}\partial_\mu V^\mu + \partial_\mu \sqrt{-g}\Gamma^\rho{}_{\rho\mu} + \sqrt{-g}\mathcal{T}_{\nu\mu}{}^\nu V^\mu\right). \tag{C.8}$$

Combining the two first terms, we have

$$\int d^3x \sqrt{-g}\ \nabla_\mu V^\mu = \int d^3x\ \left(\partial_\mu(\sqrt{-g}V^\mu) + \sqrt{-g}\mathcal{T}_{\nu\mu}{}^\nu V^\mu\right). \tag{C.9}$$

The first term vanishes just as in flat space, and we therefore are left with

$$\int d^3x \sqrt{-g}\ \nabla_\mu V^\mu = \int d^3x\ \sqrt{-g}\mathcal{T}_{\nu\mu}{}^\nu V^\mu. \tag{C.10}$$

We have therefore seen, in this appendix, that integration by parts with the covariant derivative is valid when the connection is torsion-free, but there is an additional term involving the torsion when is not.